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ERRATA

'On some remarkable types of Multiplicative Functions'

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BY

C. S. VENKATARAMAN

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ON SOME REMARKABLE TYPES OF MULTIPLICATIVE FUNCTIONS*.

BY

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[Received 6th April, 1946].

1. INTRODUCTION. In his memoir† "The Theory of Multiplicative Arithmetic Functions", Dr. R. Vaidyanathaswamy mentions certain important types of multiplicative functions of $r(>1)$ arguments, viz. (1) the ordinal, (2) the principal and (3) the modular functions.

Thus, the multiplicative function $f(M_1,..,M_r)$ is said to be an "ordinal" function if it vanishes whenever certain prescribed inequalities of the form $M_i > M_j$ hold between the arguments. M_i and M_j are called the minor and the major arguments, respectively, of the ordinal function.

$f(M_1,..,M_r)$ is called a "principal" function if it vanishes whenever any two of its arguments are unequal.

If a multiplicative function $f(M_1,..,M_r)$ has the property that its value is unaltered when M_i is increased by any multiple of M_j, it is said to be a "modular" function. M_j is called a modulus of f with respect to (or relative to) M_i.

It may be easily seen that the principal function is a particular case of the ordinal function.

Recently‡ I studied certain kinds of generalized ordinal, principal and modular functions. The object of the present paper is to show that apart from these kinds of functions, there are several other types with interesting mutual relations. The method employed here is a new

* I am indebted to Dr. R. Vaidyanathaswamy for his valuable suggestions and help in the preparation of this paper.

† *Trans. Amer. Math. Soc.* 33 (1931) 579-662. Throughout this paper, this memoir will be referred to as M.F.

‡ In a thesis submitted for the M. Sc. Degree of the University of Madras.

one, viz. that of convolution. We begin by explaining the process of convolution.*

2. CONVOLUTION. Let $f(M_1,..,M_r)$ be a multiplicative function of r (>1) arguments. Then the process of convolving M_1, M_2 in f consists in forming the function $\phi(M, M_3,..,M_r) = \Sigma f(M_1, M_2..,M_r)$, where the summation on the right side is for all values of M_1, M_2 such that $M_1M_2 = M$. ϕ will be called a 'convolute' of f and can be easily shown to be a multiplicative function of its $(r-1)$ arguments.

Also if the generating series of f to the base p is

$$f_p(x_1,..,x_r) = \sum_{m_i=0}^{\infty} a_{m_1,m_2,..,m_r} x_1^{m_1} x_2^{m_2} .. x_r^{m_r} \quad (i = 1, 2,..., r) \text{ then}$$

that of ϕ to the same base is

$$\phi(x, x_3,..,x_r) = \sum_{m_i=0}^{\infty} \Big(\sum_{m_1+m_2=m} a_{m_1, m_2,..,m_r} \Big) x^m x_3^{m_3} .. x_r^{m_r}.$$

Thus the generating series of ϕ is obtained from that of f by putting $x_1 = x_2 = x$.

The process of convolution is evidently applicable to any number S of arguments and is equivalent to the identification of the corresponding variables in the generating series. Hence in convolving several arguments, it is immaterial whether we convolve them all together or in stages.

The process of composition, viz. forming the function $f(M_1,..,M_r) = \sum_{\delta_i/M_i} f_1(M_1/\delta_1,..,M_r/\delta_r)\, f_2(\delta_1, \delta_2,..,\delta_r)$, $(i = 1,..,r)$, from $f_1(M_1,..,M_r)$ and $f_2(M_1,..,M_r)$ can be explained in terms of convolution. Thus f is obtained by convolving the r pairs $(M_i. N_i)$ in the product $f_1(M_1, M_2,..,M_r) \times f_2(N_1, N_2,..,N_r)$.

If $f(M_1,..,M_r)$ and $\phi(M_1,..,M_r)$ be two functions of the same arguments and Ω represents any convolution of

* *See* M.F. pp. 591-595.

the arguments, it can be easily shown that $\Omega[f\phi] = [\Omega f][\Omega\phi]$, where $f\phi$ denotes the composite of f and ϕ.

The composite of $f\ (M_1,..,M_r)$ and $E_0\ (M_1,..,M_i) \times E(M_{i+1},..,M_r)$ is called the integral of f with respect to $M_{i+1},..,M_r$, where E_0 and E are given by

$$E_0(M_1,..,M_r) = \begin{cases} 1, \text{ when } M_1 = M_2 = .. = M_r = 1 \\ 0, \text{ otherwise}; \end{cases}$$

and $E(M_1,..,M_r) = 1$ for all value of $M_1,..,M_r$.

3. We shall next prove certain general properties of ordinal functions. It is enough if we consider the ordinal function $f\ (M_1,..,M_r)$ which vanishes whenever an ordinal relation of the type $M_p > M_j$ holds between a single pair of arguments only, since evidently we can by repetition of the arguments deduce the properties when the ordinal relation of the above kind holds between more than one pair of arguments.

THEOREM 1. *If $f(M_1,..,M_r)$ is an ordinal function which vanishes whenever $M_i > M_j$ it also vanishes whenever M_i does not divide M_j.**

Conversely, if a multiplicative function $f(M_1,..,M_r)$ vanishes whenever M_i does not divide M_j, it is an ordinal function with M_i and M_j as the minor and the major arguments respectively.

For, $f(M_1,..,M_r) = \prod_{\lambda=1,2,..} f(p_\lambda^{a_{\lambda 1}},..,p_\lambda^{a_{\lambda r}})$ since f is multiplicative. Now $f\ (p^{m_1},..,p^{m_r})$ for any prime p vanishes whenever $p^{m_i} > p^{m_j}$ by hypothesis. Hence f vanishes when $m_i > m_j$, i.e. when a prime p occurs to a higher power in M_i than in M_j. Thus f vanishes whenever M_i does not divide M_j.

Conversely, if a multiplicative function f vanishes whenever M_i does not divide M_j, it necessarily vanishes if $M_i > M_j$, for then M_i does not divide M_j. Hence f is ordinal.

THEOREM 2. *If $f\ (M_1,..,M_r)$ is an ordinol function and $\phi\ (M_1,..,M_r)$ is any multiplicative function, then (a) their product*

* M. F. p. 656, Theorem XXXIX.

is an ordinal function, (b) their composite is an ordinal function with M_i, M_j *as the minor and major arguments, if and only if,* ϕ *vanishes whenever* $M_i > M_j$.

PROOF. (*a*) is obvious. To prove (*b*) let us first assume that ϕ is an ordinal function, vanishing whenever $M_i > M_j$. Let $F(M_1,..,M_r) = f\phi = \sum_{\delta_i/M_i} f(M_1/\delta_1,..,M_r/\delta_r)$
$\times\phi(\delta_1, \delta_2,..,\delta_r)$ $(i = 1, 2,..,r)$.

Now if $M_i/\delta_i > M_j/\delta_j$, f vanishes by hypothesis; while if $M_i/\delta_i \leqslant M_j/\delta_j$, then $\delta_i = M_i\delta_i/M_i \geqslant M_i\delta_j/M_j > M_j\delta_j/M_j = \delta_j$, i.e. $\delta_i > \delta_j$. Hence ϕ vanishes.

Thus every term on the right side vanishes whenever $M_i > M_j$ and hence F is ordinal with M_i, M_j as the minor and the major arguments.

Next let us assume that ϕ is a multiplicative function which does not vanish for all values of $M_i > M_j$, i.e. there are values of M_i, M_j such that $M_i > M_j$ and $\neq 0$ for these pairs of values.

As before let F be the composite of f and ϕ. Since ϕ is not ordinal with respect to M_i, M_j, there exist a prime p and a set of members $m_1, m_2,..,m_r$ such that $\phi(p^{m_1},..,p^{m_i}, p^{m_j},..,p^{m_r}) \neq 0$ for $m_i > m_j$. Let S_j be the smallest value of m_j for which this is true. Suppose S_i to be the corresponding smallest value of m_i, among the set of numbers $m_1,..,m_i..,m_j,..,m_r$. Now $\phi(p^{m_1},..,p^{S_i},..,p^{S_j},..,p^{m_r}) \neq 0$ and $S_i > S_j$.

Again let s_1 be the corresponding smallest value of m_1 in the set $m_1..s_i, s_j,..,m_r$. Clearly, in this manner we can fix the numbers $s_2, s_3,..,s_r$ also such that

$$\phi(p^{s_1},..p^{s_i},..p^{s_j},..,p^{s_r}) \neq 0 \text{ and } s_i > s_j \tag{1}$$

and $\phi(p^{\delta_1}, p^{\delta_i}, p^{\delta_j},..,p^{\delta_r}) = 0$ if any δ is less than the corresponding s and δ_i greater than δ_j. (2)

Now

$$F(p^{s_1},..,p^{s_i},..p^{s_j},..p^{s_r}) = \sum_{\delta_i} f\left(\frac{p^{s_1}}{p^{\delta_1}},..\frac{p^{s_i}}{p^{\delta_i}},..\frac{p^{s_j}}{p^{\delta_j}},..\frac{p^{s_r}}{p^{\delta_r}}\right)\phi(p^{\delta_1},p^{\delta_2},..,p^{\delta_r}),$$

$$\delta_i = 0, 1,..s_i, \; i = 1, 2,..r.$$

If $\delta_i \leqslant \delta_j$ then $p^{\delta_i} \leqslant p^{\delta_j}$ and therefore $\frac{p^{s_i}}{p^{\delta_i}} \geqslant \frac{p^{s_i}}{p^{\delta_j}} > \frac{p^{s_j}}{p^{\delta_j}}$. Hence f vanishes by hypothesis.

Let $\delta_i > \delta_j$. Then $\phi(p^{\delta_1}, p^{\delta_2},..,p^{\delta_i}, p^{\delta_j},..,p^{\delta_r})$ vanishes by (1) and (2), if $\delta_i \neq s_i$ $(i = 1, 2,..,r)$.

Hence the right side reduces to $\phi(p^{s_1}, p^{s_2},..,p^{s_i}, p^{s_j},p^{s_r}) \neq 0$. Thus $F \neq 0$ for $s_i > s_j$.

Hence F is not ordinal with respect to M_i, M_j.

The importance of Theorem 2 (*b*) lies in proving the ordinal nature of the inverse of f. Thus we have

THEOREM 3. *The inverse of an ordinal function $f(M_1, M_2,..,M_r)$ which vanishes whenever $M_i > M_j$, is ordinal with M_i, M_j as minor and major arguments.*

Conversely if the inverse $f^{-1}(M_1,..,M_r)$ of a multiplicative function f is ordinal with M_i, M_j as minor and major arguments, then f is necessarily an ordinal function of the same kind.

For, $f(M_1, M_2,..,M_r).f^{-1}(M_1, M_2,..,M_r) = E_0(M_1, M_2,..,M_r)$ and E_0 is obviously an ordinal function vanishing whenever $M_i > M_j$.

Hence by (*b*) of Theorem 2 it follows that f^{-1} vanishes for $M_i > M_j$.

The second part is immediate from the fact that of the functions f, f^{-1}, each is the inverse of the other.

THEOREM 4. *The integral of an ordinal function $f(M_1, M_2,..,M_r)$ with respect to the major argument M_j is an ordinal function of the same kind, while that with respect to the minor argument M_i is a modular function with M_j as modulus, relative to M_i.*

The first part follows from Theorem 2 (*b*) since $E_0(M_1,..,M_i, M_{j-1}, M_{j+1},..,M_r)E(M_j)$ is an ordinal function which vanishes for $M_i > M_j$.

To prove the second part, let F be the integral of f with respect to M_i, so that $F(M_1,..,M_r)$

$$= f\,(M_1,..,M_r) \quad E_0(M_1,..,M_{i-1}, \ M_{i+1},..,M_r) \quad E(M_i)$$
$$= \sum f(\delta_1,..,\delta_i,\ \delta_j,..,\delta_r)\ E_0\left(\frac{M_1}{\delta_1},..\frac{M_{i-1}}{\delta_{i-1}},..\frac{M_r}{\delta_r}\right) E\left(\frac{M_i}{\delta_i}\right)$$
$$= \sum_{\delta_i \mid M_i} f(M_1,\ M_2,..,\delta_i,\ M_r).$$

Now, by Theorem 1, f vanishes if δ_i does not divide M_j. Hence δ_i is a common factor of M_i, and M_j, i.e. a divisor of g_{ij} the greatest common divisor of M_i and M_j.

Thus, $F(M_1,..,M_r) = \sum_{\delta_i \mid g_{ij}} f(M_1,..,\delta_i,\ M_r)$

$$= F(M_1,\ M_2,..,g_{ij},..M_r).$$

Since $(M_i,\ M_j) = (M_i + \lambda M_j,\ M_j) = g_{ij}$, it follows that F has the modulus M_j with respect to M_i and that the value of F depends on M_i only through its g.c.d. with M_j.

We shall now prove the converse of the second part of Theorem 4, thereby establishing the fundamental mutual relation between the ordinal and the modular functions.

THEOREM 5. *If $F(M_1,..M_i,..,M_j,\ M_r)$ is a modular function with the modulus M_j relative to M_i, then its composite with $E_0(M_1,..,M_{i-1}, M_{i+1},..,M_r)\ E^{-1}(M_i)$ is an ordinal function which vanishes whenever $M_i > M_j$.*

PROOF. Let $f(M_1,..,M_i,..,M_j,..,M_r)$

$$= F(M_1,..,\ M_i,..,M_j,\ M_r)\ E_0(M_1,..,M_r)\ E^{-1}(M_i)$$
$$= \sum_{\substack{\delta_i \mid M_i \\ (i=1,2,..,r)}} F\left(\frac{M_1}{\delta_1},..\frac{M_i}{\delta_i},..\frac{M_r}{\delta_r}\right) E_0(\delta_1,..,\delta_{i-1},\delta_{i+1},\delta_r) E^{-1}(\delta_i)$$
$$= \sum_{\delta_j \mid M_i} F(M_1,..,\ M_i/\delta_i\ ,..,M_r)\ \mu(\delta_i). \qquad (1)$$

[The inverse of $E(M)$, viz. $E^{-1}(M)$ can be easily seen to be the Möbius function $\mu(M)$.]

Now

$$f(M_1,\ldots,M_i,\ldots,M_r)=\prod_{(\lambda=1,2,\ldots)} f(p_\lambda^{a_{\lambda 1}},\ldots,p_\lambda^{a_{\lambda i}},\ldots,p_\lambda^{a_{\lambda r}}), \quad (2)$$

where

$$M_i=p_1^{a_{1i}}p_2^{a_{2i}}p_3^{a_{3i}}\ldots(i=1,2,\ldots r)$$
$$p_1<p_2<p_3<\ldots$$

and for any prime p,

$$f(p^{m_1},\ldots,p^{m_i},\ldots,p^{m_j},\ldots,p^{m_r})=\sum_{\delta_i/p^{m_i}} F(p^{m_1},\ldots\frac{p^{m_i}}{\delta_i},\ldots,p^{m_r})\mu(\delta_i)$$

$$=F(p^{m_1},\ldots p^{m_i},p^{m_{i+1}},\ldots p^{m_r})-F(p^{m_1},\ldots p^{m_i-1},p^{m_{i+1}},\ldots p^{m_r}). \quad (3)$$

Since F is a modular function with M_j as modulus relative to M_i,

$$F(p^{m_1},\ldots,p^{m_i},p^{m_{i+1}},\ldots,p^{m_j}\ldots,p^{m_r})$$
$$=F(p^{m_1},\ldots,p^{m_j},p^{m_{i+1}}\ldots,p^{m_j},p^{m_r})$$

whenever $m_i>m_j$. For let $m_i=m_j+k$ $(k\geqslant 1)$. Then

$$F(p^{m_1},\ldots,p^{m_i},\ldots,p^{m_r})$$
$$=F(p^{m_1},\ldots,p^{m_j+k},p^{m_{i+1}},\ldots,p^{m_j},\ldots,p^{m_r})$$
$$=F(p^{m_1},\ldots,p^{m_j}+\lambda p^{m_j},p^{m_{i+1}},\ldots,p^{m_j},\ldots,p^{m_r}),$$

where $\lambda=p^k-1(>0)$

$$=F(p^{m_1},\ldots,p^{m_j},p^{m_{i+1}},\ldots,p^{m_j},\ldots,p^{m_r}) \quad (4)$$

on account of the modular property.

Now if $m_i>m_j$, then $m_i-1\geqslant m_j$. Then from (3) and (4), it follows that $f(p^{m_1},\ldots,p^{m_i},\ldots p^{m_j}\ldots,p^{m_r})=0$ whenever $p^{m_i}>p^{m_j}$.

Therefore $f(M_1,\ldots,M_i,\ldots,M_j,\ldots M_r)=0$ if M_i does not divide M_j and is thus an ordinal function with M_i, M_j as the minor and the major arguments respectively.

Note. In arriving at the result (4), we have incidentally shown that if $F(M_1,\ldots,M_i,\ldots,M_j,\ldots,M_r)$ is a modular function with M_j as modulus relative to M_i, then $F(M_1,\ldots, M_i\ldots, M_j,\ldots M_r)=F(M_1,\ldots,g_{ij},\ldots M_j,\ldots M_r)$ where $g_{ij}=\prod_{(\lambda=1,2,\ldots)} p_\lambda^{min\,(a_{\lambda i},a_{\lambda j})}$, the g.c.d. of M_i and M_j. This is a fundamental property of modular functions.

3.1. REMARKS. The theorem proved above for ordinal functions in which an ordinal relation holds between a single pair of arguments only, can be clearly extended to cases where an ordinal relation holds between several pairs of arguments. We shall state the results analogous to Theorem 4, for the following typical kinds of ordinal functions, viz. (1) $f(M_1, M_2, .., M_r, N_1, .., N_s)$ in which an ordinal relation holds between k distinct pairs of arguments N_i, M_i, N_i's being the minor arguments, i.e. f vanishes whenever $N_i > M_i$ $(i = 1, 2, .., k)$ and $k \leqslant$ the smaller of r, s, and (2) $\psi(M_1, M_2, .., M_r, N_1, .., N_s)$ in which an ordinal relation holds between two groups of arguments N_i, M_j such that each argument of the first group (the group of N's) is a minor argument relative to each argument of the second group (the group of M_j's); in other words ψ vanishes whenever any $N_i >$ any M_j $(i = 1, 2, .., k \leqslant s$; and $j = 1, 2, .., l \leqslant r)$.

It would be convenient to refer to f and ψ, as "ordinal functions of the first kind, of type k" and "ordinal functions of the second kind, of type l, k," respectively.

The analogous theorems are:—

(*a*) *The integral of an ordinal function of the first kind, of type k, with respect to its k minor arguments* N_i $(i = 1, 2, .., k)$, *is a modular function with the moduli* M_i *relative to* N_i $(i = 1, 2, .., k)$ *(which may be called modular function of the first kind, of type k).*

(*b*) *The integral of an ordinal function of the second kind, of type l, k, with respect to the group of k minor arguments* N_i *is a modular function with each* M_j *as modulus relative to each* N_i *(we shall refer to this integral as modular function of the second kind, of type l, k).*

4. We now proceed to study the result of convolution of ordinal functions. We shall concern ourselves

with the two typical ordinal functions mentioned in the last para.

THEOREM 6. *The convolute of an ordinal function of the first kind and of type k, (viz. the function formed by convolving separately the k minor arguments and the k major arguments) is an ordinal function of the first kind of type* 1.

Let the convolute of f be

$$\psi(M, M_{k+1}, .., M_r, N, N_{k+1}, .., N_s) = \Sigma f(M_1, .., M_r, N_1, .., N_s),$$

where $$M = \prod_{i=1}^{k} M_i,\ N = \prod_{i=1}^{k} N_i.$$

Suppose $N > M$. The N_i cannot be less than M_i for $i = 1, 2, .., k$, simultaneously. Therefore there is at least one value of i (say α) $\leqslant k$ such that $N_\alpha \geqslant M_\alpha$. If $N_\alpha = M_\alpha$, then $\frac{N}{N_\alpha} > \frac{M}{M_\alpha}$ and by a similar reasoning, it can be shown that there is a $\beta \neq \alpha$ for which $N_\beta \geqslant M_\beta$ $(\beta \leqslant k)$. Again if N_β happens to be equal to M_β, we can repeat the argument so that for every set of numbers $N_1, N_2, .., N_k = N > M = M_1, M_2, .., M_k$, there is a value of i, such that $N_i > M_i$ for $i \leqslant k$.

Hence every term on the right side and therefore ψ vanishes if $N > M$. Thus ψ is an ordinal function of the first kind and of type 1.

THEOREM 7. *The convolute of the integral with respect to the k minor arguments of an ordinal function of the first kind and of type k (viz. the result of convolving separately all the minor arguments and the major arguments) is a modular function of the first kind of type* 1.

For if Ω represents the convolution of $N_1, .., N_k$ and $M_1, .., M_k$, and

$$F(M, M_{k+1}, .., M_r, N, N_{k+1}, .., N_s)$$

2

$$= \Omega[f(M_1,..,M_r, N_1,..,N_s)$$
$$\times E_0(M_1,..,M_r, N_{k+1}, N_s)\ E(N_1,..N_k)]$$

then $F = \Omega f(M_1,..,M_r, N_1,..,N_s)$

$$\times \Omega E_0(M_1,..,M_r, N_{k+1},..,N_s)\ E(N_1,..,N_k)$$
$$= \psi(M, M_{k+1},..,M_r, N, N_{k+1},..,N_s)$$
$$\times E_0(M, M_{k+1},..M_r, N_{k+1},..,N_s)\ E(N).$$

[It can be seen that $E_0(M, M_{k+1},..M_r, N_{k+1},..,N_s)\ E(N)$ is the convolute of $E_0(M_1,..M_r, N_{k+1},..,N_s)\ E(N_1,..,N_k)$.]

Hence by Theorem 4, it follows that F is a modular function with modulus M relative to N, i.e. F is a modular function of the first kind, of type 1.

COR. *The convolute of a modular function of the first kind, of type k (viz. the function formed by convolving separately the k modular arguments and the k non-modular arguments) is a modular function of the first kind, of type* 1.

For F is a modular function of the first kind, of type k by 3.1 (a), and by Theorem 5 any modular function of the first kind of type k is the integral of an ordinal function of the first kind of type k, with respect to the minor arguments.

THEOREM 8. *The convolute of an ordinal function of the second kind of type l, k (viz. the function formed by convolving separately the group of k minor arguments and the group of l major arguments) is an "indexed ordinal function."*

PROOF. Let $\psi(M, M_{l+1},..,M_r, N, N_{k+1},..,N_s)$

$$= \Sigma f(M_1,..,M_r, N_1,..,N_s),$$

where $\quad M = M_1...M_l,\ N = N_1...N_k.$

Now f vanishes if $N_i > M_j$ $(i = 1, 2,..,k,\ j = 1,..,l)$
(i.e. if $N_1 > M_1, N_2 > M_1,..,N_k > M_1$ or if $N > M_1^k$
also if $N_1 > M_2, N_2 > M_2,..,N_k > M_2$ or if $N > M_2^k$

. . . .

. . . .

if $N_1 > M_l, N_2 > M_l,..,N_k > M_l$ or if $N > M_l^k$
i.e. if $N_l > M^k$.

Thus f and therefore every term on the right side vanishes whenever $N^l > M^k$.

Hence ψ is a new type of ordinal function vanishing whenever $N^l > M^k$ and it is appropriate to refer to it as an indexed ordinal function.

NOTE. ψ vanishes whenever N^l does not divide M^k, for then there is a pair of values i, j, such that N_i does not divide M_j corresponding to every set of numbers $N_1 N_2 .. N_k = N$ and $M_1 ... M_l = M$ and therefore f vanishes by Theorem 1.

The following special cases are worthy of mention:—

(1) If $l = k$, then ψ reduces to an ordinal function of the first kind of type 1.

(2) If $l = r$, $k = s$, then ψ becomes an indexed ordinal function of two arguments.

(3) If f is a principal function, then considering N_i's first as minor arguments and then as major arguments, it follows by (2) that $\psi(M, N) = 0$ if $N^r >$ or $< M^s$.

ψ may be called a quasi-principal function which vanishes if $N^r \neq M^s$.

Lastly we shall establish the property of the convolute of the integral of an ordinal function of the second type l, k.

THEOREM 9. *The convolute of the integral with respect to the group of the k minor arguments of an ordinal function of the second type l, k is an indexed modular function.*

PROOF. Let Ω represent the convolution of $M_1, .., M_l$ and $N_1, .., N_k$ and let

$$F(M, M_{l+1}, .., M_r, N, N_{k+1}, .., N_s) = \Omega[f(M_1, .., M_r, N_1, .., N_s) \times E_0(M_1, .., N_{k+1}, N_s), E(N_1, .. N_k)]$$

$$= \Omega f(M_1, \ldots, M_r, N_1, \ldots, N_s)$$
$$\times \Omega E_0(M_1, \ldots, N_{k+1}, \ldots, N_s)\, E(N_1, \ldots, N_k)$$
$$= \psi(M, M_{l+1}, \ldots, M_r, N, N_{k+1}, \ldots, N_s).$$
$$\times E_0(M, M_{l+1}, \ldots, N_{k+1}, \ldots, N_s)\, E(N)$$
$$= \sum \psi(d, d_{l+1}, \ldots, d_r, \delta, \ldots, \delta_s)\, E_0\left(\frac{M}{d}, \ldots, \frac{N_s}{\delta_s}\right) E\left(\frac{N}{\delta}\right)$$

d/M, δ/N, d_j/M_j $(j = l+1, \ldots, r)$; δ_i/N_i $(i = k+1, \ldots, s)$,

$$= \sum \psi(M, M_{l+1} \ldots M_r, \delta, \ldots N_s)$$

$$= \sum_{\delta/N,\, \delta^l/M^k} \psi(M, \ldots, M_{l+1}, \ldots M_r, \delta, N_{k+1}, N_s).$$

Let $M = p_1^{m_1} p_2^{m_2} \ldots p_t^{m_t}$. Then $M^k = p_1^{m_1 k} p_2^{m_2 k} p_t^{m_t k}$. Therefore $M_{lk} = p_1^{\left[\frac{m_1 k}{l}\right]} p_2^{\left[\frac{m_2 k}{l}\right]} \ldots p_t^{\left[\frac{m_t k}{l}\right]}$, where M_{lk} is the greatest integer whose lth power divides M^k. If $g_{lk} = (M_{lk}, N)$, it can be easily seen that if $\delta | N$, and $\delta^l | M^k$, then $\delta | g_{lk}$. Conversely any divisor δ of g_{lk} is such that $\delta | N$, and $\delta^l | M^k$. Hence $F(M, M_{l+1}, \ldots, M_r, N, \ldots, N_{k+1}, N_s)$

$$= \sum_{\delta / g_{lk}} \psi(M, M_{l+1}, \ldots, M_r, \delta, N_{k+1}, \ldots, N_s).$$

Since $(M_{lk}, N) = (M_{lk}, N + \lambda M_{lk}) = g_{lk}$, where λ is any integer, it follows that F has the effective modulus M_{lk}.

Thus we have the concept of what may be called an *indexed modular function.*

COR. (1) *The convolute of a modular function of the second kind of type* l, k *(viz. the result of convolving separately the group of modulus and the group of non-modulus arguments) is an indexed modular function. (Sec.* 3.1 *(b) and Th.* 5).

(2) *If* $l = r$, $k = s$, F *reduces to an indexed modular function of two arguments.*

STABILITY OF RECTILINEAR PLATES.

BY

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The stability of a rectangular plate has been the subject of a number of investigations. Bryan(1) has given the solution when the plate is supported at its edges. Southwell, Skan(2) and Dean(3) have also discussed the problem in a number of papers. When the edges of the plate are clamped, approximate solutions have been obtained by Sezawa,(4) Timoshenko,(5) Taylor,(6) Trefftz,(7) Weinstein,(8) Faxen,(9) Levy(10) and others. Recently Green(11) has obtained some of these results more directly by using the double Fourier series. We shall discuss the stability of rectilinear plates with supported edges under equal thrusts in two directions, and shall give some exact solutions for triangular plates.

Let a plane isotropic sheet be subjected to stresses P_1, P_2 parallel to rectangular axes, and let these stresses be such that they can maintain a small tranverse displacement w in the plate so contracted. The differential equation satisfied by w is known to be(12)

$$D\nabla^4 w + \left(P_1\frac{\partial^2}{\partial x^2} + P_2\frac{\partial^2}{\partial y^2}\right) w = 0, \qquad (1)$$

where

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \text{ and } D = \tfrac{1}{12}\frac{Eh^2}{1-\sigma^2},$$

$2h$ being the thickness of the plate and E and σ having their usual meanings.

If we put $P_1 = P_2 = P$ (say), (1) can be written

$$\nabla^2(D\nabla^2 w + Pw) = 0. \qquad (2)$$

For rectilinear plates supported at their edges the boundary condition can be written in the form

$$\nabla^2 w = 0,\ w = 0. \qquad (3)$$

Thus if we can determine the solutions of

$$D\nabla^2 w + Pw = 0, \tag{4}$$

which satisfy the boundary condition $w=0$ we can use these very solutions for discussing the stability of supported plates. The problem of the stability of a rectilinear plate supported at its edges is therefore reduced to that of the transverse vibrations of a rectilinear membrane whose boundary is the same as that of the plate.

For the transverse vibrations of a rectangular membrane we have

$$w = W.\ \sin\frac{m\pi(x+a)}{2a}\sin\frac{n\pi(y+b)}{2b}, \tag{5}$$

the edges being $x=\pm a$, $y=\pm b$, m and n being integers. This gives

$$P = \tfrac{1}{4}D\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right), \tag{6}$$

which is what we get from Bryan's result.(1) For the stability of a square plate of side $2a$, P may not exceed $\frac{1}{2}D\pi^2/a^2$. It should be noted that this critical value of P corresponds to the gravest mode of vibration of the membrane.

For an isosceles right angled plate whose edges are $x=a, y=\pm x$, we have(13)

$$w = W\left[\cos\frac{(2m+1)\pi x}{2a}\cos\frac{(2n+1)\pi y}{2a} - \cos\frac{(2n+1)\pi x}{2a}\cos\frac{(2m+1)\pi y}{2a}\right], \tag{7.1}$$

$$P = \tfrac{1}{4}D\frac{\pi^2}{a^2}\left[(2m+1)^2+(2n+1)^2\right], \tag{7.2}$$

$$P_c = \tfrac{5}{2}\frac{D\pi^2}{a^2}, \tag{7.3}$$

P_c being the critical value of P.

For an equilateral plate with sides $x=a, y\sqrt{3}=\pm x$, we have(13)

$$v = W\Big[\sin\frac{(m-n)\pi x}{a}\cos\frac{(m+n)\pi y\sqrt{3}}{a}$$

$$-\sin\frac{(2m+n)\pi x}{a}\cos\frac{n\pi y\sqrt{3}}{a}+\sin\frac{(2n+m)\pi x}{a}\cos\frac{m\pi y\sqrt{3}}{a}\Big], \quad (8.1)$$

$$P = 4D\frac{\pi^2}{a^2}(m^2+mn+n^2), \quad (8.2)$$

$$P_c = 4D\frac{\pi^2}{a^2}. \quad (8.3)$$

For a right angled triangle containing an angle of 60° whose sides are $y=a/\sqrt{3}$, $x=a$, $y=x\sqrt{3}$, we get[13]

$$w = W\Big[\sin\frac{(m-n)\pi x}{a}\sin\frac{(m+n+1)\pi y\sqrt{3}}{a}$$

$$-\cos\left(2m+1+\frac{2n+1}{a}\right)\frac{\pi x}{a}\cos\frac{(2n+1)\pi y\sqrt{3}}{2a}.$$

$$+\cos\left(2n+1+\frac{2m+1}{2}\right)\frac{\pi x}{a}\cos\frac{(2m+1)\pi y\sqrt{3}}{2a}\Big], \quad (9.1)$$

$$P = \frac{D\pi^2}{a^2}\left[(2m+1)^2+(2m+1)(2n+1)+(2n+1)^2\right], \quad (9.2)$$

$$P_c > \frac{D\pi^2}{a^2}\cdot \quad (9.3)$$

Thus, if $(p/2\pi)$ be the frequency of the gravest mode of vibration of a rectilinear membrane, we have

$$\frac{P_c}{D} = \frac{p^2}{c^2}, \quad (10)$$

c being the ratio of the uniform tension and the superficial density ρ in the problem of the membrane.

Thus, even though the solution in (9) holds good for the symmetrical vibrations of an isosceles triangular membrane of vertical angle 120°, it cannot be used to determine P_c, as it does not give the frequency of the gravest mode. In such cases Rayleigh's method can be used to get the value of p.[14]

If A be the area of the plate we get the following comparison for the values of P_c.

Value of	Square.	Triangle. $\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi$	Triangle. $\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi$	Triangle. $\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{6}\pi$
$AP_c/D\pi^2$	2	2.31	2.50	2.69

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ON THE TRANSLATION NUMBERS OF INTEGRAL FUNGTIONS.*

BY

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1. INTRODUCTION. The object of this paper is to study the relation

$$f(z+\lambda)=g(z)f(z) \qquad (1)$$

from two points of view, $f(z)$ and $g(z)$ being integral functions. In section I, I study the properties of integral functions $f(z)$ possessing one or more numbers λ satisfying an equation of form (1). In section II, I consider the solution of the functional equation (1) for a given λ and $g(z)$. It is known† that equation (1) always admits a meromorphic solution. But this is no longer true if we require that the solution of (1) should be an integral function. For instance, if $g(z)=\sin \pi z/\lambda$, equation (1) cannot have any solution in integral functions except the trivial solution $f(z)\equiv 0$. I consider also the conditions on $g(z)$ for the existence of a non-trivial solution in integral functions of the equation (1). The problem regarding the order of the solution is also discussed.

2. DEFINITIONS. Let $f(z)$ be any integral function. A number λ for which there exists an integral function $g(z)$ such that (1) holds, will be called a *translation number* of the function $f(z)$ provided $f(z)$ has *at least one zero*. This last restriction is to avoid the trivial case when for every number λ, the relation (1) holds. If λ and $-\lambda$ are both translation numbers of $f(z)$, we shall say that λ is a

* An abstract of this paper was communicated to the Fourteenth Conference of the Indian Mathematical Society held at Delhi in December 1945.

† J. M. Whittaker, *Interpolatory Function Theory* (Cambridge Tracts, No. 33), p. 31.

a *translation number of the first kind.* Otherwise λ will be called a *translation number of the second kind.* It is evident that if λ is a translation number, then so is $m\lambda$, where m is any *positive* integer. If λ is of the first kind, then $\pm m\lambda$ are also translation numbers. We shall prove below that there can be at most two independent translation numbers of the first kind while there can be an enumerable set of independent numbers of the second kind.

SECTION I.

3. TRANSLATION NUMBERS OF THE FIRST KIND. The main properties of translation numbers of the first kind are similar to those of periods of analytic functions. The following results, though elementary are proved for the sake of completeness.

THEOREM 1. *All translation numbers of the first kind on a straight line through $z = 0$ are of the form $\pm m\lambda$, where m is a positive integer.*

PROOF. The translation numbers of any integral function cannot have finite limit points. Hence on any straight line through the origin there is a translation number λ (if there be one at all) such that $|\lambda|$ is the lower bound of the moduli of translation numbers of $f(z)$ on this line. If μ is any translation number on this line, we can write $\mu = m\lambda + \lambda_1$, where m is an integer and $|\lambda_1| < |\lambda|$. Since λ is of the first kind, it follows that $\lambda_1 = \mu - m\lambda$ is also one translation number which is impossible unless $\lambda_1 = 0$. This proves the theorem.

3.1. THEOREM 2. *If $f(z)$ possesses two translation numbers of the first kind whose ratio is not real, then there exist two translation numbers λ_1 and λ_2 such that every translation number (whether of the first or second kind) is of the form $m_1\lambda_1 + m_2\lambda_2$, where m_1 and m_2 are integers. Moreover all translation numbers are of the first kind.*

PROOF. Since the translation numbers of an integral unction cannot have finite limit points, we can find a

translation number λ_1 having the least modulus and of the first kind. Among the remaining translation numbers of the first kind let λ_2 denote the one with the least modulus such that the vector $(0, \lambda_2)$ makes numerically the least angle (which is not zero, by hypothesis) with the vector $(0, \lambda_1)$. It is obviously enough to prove that any translation number λ whether of the first or second kind is of the form given in the theorem. We can write $\lambda = x_1\lambda_1 + x_2\lambda_2$, where x_1 and x_2 are real. If x_1 or x_2 or both be irrational we can find (by known properties of irrational numbers) an infinite number of triplets of integers (p, m_1, m_2) with $p > 0$ such that the set of numbers $\{p\lambda + m_1\lambda_1 + m_2\lambda_2 = (px_1 + m_1)\lambda_1 + (px_2 + m_2)\lambda_2\}$ has finite limit points, which is impossible since all these are translation numbers. If x_1 and x_2 be both rational, we can find a triplet (p_1, m_1, m_2) such that $p\lambda = m_1\lambda_1 + m_2\lambda_2$ so that $p\lambda$ and $-p\lambda$ are both translation numbers and hence $-p\lambda + (p-1)\lambda = -\lambda$ is also a translation number. Hence λ is of the first kind. If x_1 or x_2 be an integer the result follows from the previous theorem. Let both be rational non-integers. Then we can find θ_1 and θ_2 such that $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and such that $\theta_1\lambda_1 + \theta_2\lambda_2$ is a translation number of the first kind. So $(1-\theta_1)\lambda_1 + (1-\theta_2)\lambda_2$ is also one of the same kind. Both these lie in the parallelogram whose adjacent sides are the vectors $(0, \lambda_1)$, $(0, \lambda_2)$. Also $|\theta_1\lambda_1 + \theta_2\lambda_2| < (\theta_1 + \theta_2)|\lambda_2|$ and $|(1-\theta_1)\lambda_1 + (1-\theta_2)\lambda_2| < (2-\theta_1-\theta_2)|\lambda_2|$ and one of these gives a contradiction in view of the choice of the numbers λ_1 and λ_2. Hence x_1 and x_2 are integers and our theorem is proved.

3.2. THEOREM 3. *A necessary and sufficient condition that λ should be a translation number of the first kind for $f(z)$ is that $g(z)$ in (1) should have no zeros. If $\{z_n\}$ denotes the zeros of $f(z)$, this statement is equivalent to the condition that the two sequences $\{z_n\}$ and $\{z_n + \lambda\}$ should be the same.*

PROOF. From (1), λ is a translation number of the first kind for $f(z)$ if and only if both $f(z+\lambda)/f(z)$ and

$f(z-\lambda)/f(z)$ are integral functions. This is equivalent to saying that both $f(z-\lambda)/f(z) = g(z)$ and $f(z)/f(z+\lambda) = 1/g(z)$ are integral functions which is possible if and only if $g(z)$ has no zeros. The second part of the theorem is obvious from the first part.

3·3. THEOREM 4. (*i*) *If* λ *is a translation number of the first kind for* $f(z)$ *then* $f(z) = e^{h(z)}\,\theta(z)$, *where* $h(z)$ *is an integral function and* $\theta(z)$ *is an integral function of period* λ. *The converse of this is also true.*

(*ii*) *Let* $f(z)$ *have two translation numbers of the first kind whose ratio is not real. Let* λ_1 *and* λ_2 *(which will be called a pair of primitive translation numbers) be the two translation numbers of the first kind as described in Theorem* 2. *Let* $\sigma(z)$ *denote the canonical product associated with the lattice* $\{m_1\lambda_1+m_2\lambda_2\}$, $m_1, m_2 = 0, \pm 1, \pm 2 \ldots$ *Then there exists a finite number of points* $(z_1, z_2, \ldots, z_p)$ *such that* $f(z) = e^{h(z)} \prod_{i=1}^{p} \sigma(z-z_i)$. *The converse is also true.*

PROOF OF (*i*). Since the sequence $\{z_n\}$ of zeros of $f(z)$ is invariant for the transformation $z' = z \pm \lambda$, it follows* that there exists an integral function $\theta(z)$ with period λ and having the zeros $\{z_n\}$. Hence $f(z)$ is of the form given in the theorem. The converse is obvious.

PROOF OF (*ii*). Let $z_1, z_2, \ldots, z_p$ denote the incongruent zeros (counted with their proper multiplicity) of $f(z)$ in the parallelogram whose adjecent sides are the vectors $(0, \lambda_1)$ and $(0, \lambda_2)$. Then it is easily verified that $\prod_{i=1}^{p} \sigma(z-z_i)$ has precisely the same zeros as $f(z)$. Hence the result. Here also the converse is obvious.

3·4. REMARK. If $f(z)$ is of order ρ in Theorem 4 (*ii*), it follows that $\rho \geqslant 2$ and $h(z)$ is a polynomial of

* See my paper on 'Periodic Integral Functions' *J.I.M.S.*, (2) 5 (1941), 1-17.

degree not exceeding ρ. Since $\prod_{i=1}^{p} \sigma(z-z_i)$ is of order two, it follows that ρ must be an integer. The function $\sigma(z)$ is the usual Weierstrassian σ-function. Since it is known that $\sigma(z+\lambda)=e^{az+b}\,\sigma(z)$ (which can also be proved directly by using the oddness of $\sigma(z)$) it follows that $f(z+\lambda)=e^{P(z)}f(z)$, where $P(z)$ is of degree $\rho-1$ at most.

4. TRANSLATION NUMBERS OF THE SECOND KIND. The properties of translation numbers of the second kind are not so simple. The following theorem is obvious from definition.

THEOREM 5. *A necessary and sufficient condition that λ should be a translation number of the second kind for $f(z)$ is that the set $\{z_n+\lambda\}$ should be a proper subset of the sequence $\{z_n\}$ of zeros of $f(z)$.*

4.1. As already remarked, if λ is a translation number of the second kind for $f(z)$ then so are $m\lambda$, $m=1, 2, \ldots$(but not $-m\lambda$). We shall now examine the number of independent translation numbers of the second kind which an integral function can possess. Obviously the set $T(f)$ of independant translation numbers of the second kind cannot possess finite limit points. So the set $T(f)$ is either finite or is a sequence tending to infinity. It will turn out that both can happen. The following theorem is obvious from Theorem 5.

THEOREM 6. (*i*) *In order that a sequence $\{\lambda_n\}$ should be translation numbers of the second kind for some integral function, it is necessary and sufficient that the set $\{m_1\lambda_1+m_2\lambda+\ldots+m_p\lambda_p\}$, where $m_i \geqslant 0$ are integers and $p=1, 2, \ldots$ should have infinity as its sole limit point.*

(*ii*) *In order that a finite set $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ should be the translation numbers of the second kind for some integral function, it is necessary and sufficient that the set $\{m_1\lambda_1+m_2\lambda_2+\ldots+$*

$m_p\lambda_p \}$, $m_i \geqslant 0$ *being integers, should have infinity as its sole limit point.*

4.2. THEOREM 7. *Let λ be a translation number of the second kind. Then there cannot exist another translation number μ (whether of the first or second kind) such that λ/μ is real and negative.*

PROOF. If μ is of the first kind, then $-\mu$ is also a translation number. Hence it is enough to prove the theorem under the assumption that μ is of the second kind. Let $\lambda/\mu = -1/x$, $x > 0$. Then $\mu = -\lambda x$. Suppose x is irrational. Then $m\mu + n\lambda = (-mx+n)\lambda$ is also a translation number for all positive integral values of m and n. But it is known that the set $\{ -mx+n \}$ has every point as a limit point. So x cannot be irrational. Let $x = p/q$, where p and q are positive integers so that $\lambda/(-q) = \mu/(+p) = a\lambda + b\mu$, where a and b are positive integers such that $bp - aq = 1$. Hence $\lambda = -q(a\lambda + b\mu)$ is a translation number so that $-aq\lambda$ and so $-\lambda$ is also a translation number which is contrary to hypothesis. So the theorem is proved.

4.3. THEOREM 8. *Let λ_1 and λ_2 be two translation numbers of the second kind such that λ_1/λ_2 is not real. Then no number of the form $\lambda = -x_1\lambda_1 - x_2\lambda_2$, $x_1 \geqslant 0$, $x_2 \geqslant 0$ can be a translation number.*

PROOF. The cases $x_1 > 0$, $x_2 = 0$ and $x_1 = 0$, $x_2 > 0$ reduce to the previous theorem. So we shall suppose that both x_1 and $x_2 > 0$. If one or both of x_1 and x_2 be irrational then the set $\{ p\lambda + m_1\lambda_1 + m_2\lambda_2 \}$, p, m_1, m_2 being *positive* integers, will have finite limit points and this is impossible since each such number is a translation number. Let x_1 and x_2 be both rational. Then we can find positive integers p, m_1 and m_2 such that $p\lambda = -m_1\lambda_1 - m_2\lambda_2$. Hence $-m_2\lambda_2 = p\lambda + m_1\lambda_1$ is also a translation number so that $-\lambda_2 = -m_2\lambda_2 + (m_2 - 1)\lambda_2$ is also a translation number which is contrary to hypothesis.

4.4. REMARK 1. Similar arguments prove that if λ_1 is of the first kind and λ_2 of the second kind then no number of the form $x_1\lambda_1 - x_2\lambda_2$ with $x_2 > 0$ can be a translation number.

REMARK 2. It can now be seen that subject to the restrictions imposed by Theorems 7 and 8, the set $T(f)$ can be any finite set or set having its sole limit point at infinity. Using Theorem 6 it is easily seen that the following sets can be independent translation numbers of integral functions.

(i) $\{\sqrt{p_n}\}$, where $p_1 = 1, p_2, p_3, \ldots$ being the sequence of prime positive integers, (Theorem 7).

(ii) $1, i, \sqrt{2}(1+i), \sqrt{2}(1-i), -\sqrt{2}(1-i)$, (Theorem 8).

(iii) $1,\ i,\ \sqrt{p_n},\ \sqrt{p_n}+i,\ \sqrt{p_n}-i,\ n = 1, 2, \ldots$ where p_1, $p_2, \ldots$ are the primes, (Theorem 8).

SECTION II.

5. We shall now study the equation (1) as a functional equation. Firstly we have

THEOREM 9. *Let $g(z)$ in (1) be an integral function without zeros. Then (1) has always a solution $f(z)$. Any other solution of (1) will be of the form $f(z)\phi(z)$, where $\phi(z)$ is an integral function with period λ.*

PROOF. Let $g(z) = e^{h(z)}$. We always find* a function $P(z)$ such that $P(z+\lambda) - P(z) = h(z)$. The function $f(z) = e^{P(z)}$ is solution of (1). If $F(z)$ be any other solution then $F(z)/f(z)$ must have period λ and must be an integral function since $f(z)$ has no zeros. Hence $F(z) = \phi(z)f(z)$ as stated in the theorem.

5.1. REMARK. If $g(z)$ is of finite order ρ in Theorem 9, then ρ is an integer and $h(z)$ is a polynomial of degree ρ. Hence $P(z)$ can be chosen to be a polynomial of degree $\rho+1$. Hence $f(z)$ is of order $\rho+1$. Since the

* J. M. Whittaker, *l. c.* p. 25.

only periodic integral function of finite order without zeros is of order one and is of the form $Ae^{2\pi iz/\lambda}$ it follows that (1) cannot have a solution of order less than $1+\rho$ and the solution $f(z)$ without zeros and of finite order is unique except for a factor of the form $Ae^{2\pi iz/\lambda}$.

6. Next we consider the case when $g(z)$ in (1) has zeros. For the convenience of stating the results in a simple manner we shall suppose that $\lambda > 0$ in (1). We shall denote by $R(z)$ and $I(z)$ the real and imaginary parts of z.

THEOREM 10. *Let $\{z_n\}$ denote the zeros of $g(z)$ in (1). Then a necessary and sufficient condition that (1) should have an integral function solution $f(z)$ (other than $f(z)\equiv 0$) is that the set $\{z_n+m\lambda\}$, m, $n=1, 2, \ldots$ should have no finite limit point.*

PROOF. The condition is necessary because if $f(z)$ is a solution of (1), the set mentioned in the theorem consists of the zeros of $f(z)$. Now suppose the condition is satisfied. Let $f(z)$ denote the canonical product with its zeros at $z_n+m\lambda$, $m, n=1, 2, \ldots$. Then the zeros of $f(z+\lambda)$ will be $\{z_n+(m-1)\lambda\}$, $m, n=1, 2, \ldots$. Hence $f(z+\lambda)/f(z)$ is an integral function having the same zeros $\{z_n\}$ as $g(z)$. Here $f(z+\lambda)=e^{h(z)}\,g(z)\,f(z)$. Choose $P(z)$ so that $P(z+\lambda)-P(z)=-h(z)$. Then $F(z)=e^{P(z)}\times f(z)$ is a solution of (1).

6.1. In the statement of the following theorem we agree that if a sequence $\{z_n\}$ contains a particular number α repeated *infinitely often*, then α *is a limit point* of the sequence.

THEOREM 11. *Let $\{z_n\}$ denote the zeros of $g(z)$ in (1) where $\lambda > 0$. Then a necessary and sufficient condition that (1) should have a non-trivial integral function solution is that the set $\{I(z_n)\}$ for which $R(z_n) < 0$ should have no finite limit point.*

PROOF. THE CONDITION IS SUFFICIENT. By Theorem 10, it is enough to prove that the number of points of the sequence $E = \{z_n + m\lambda\}$, $m, n = 1, 2, \ldots$ in any circle C_r: $|z| \leqslant r$, is finite. [It is to be noted that any point $z = z_m + m\lambda$ of E is to be counted as many times as it could be obtained starting from the zeros of $g(z)$ and adding positive integral multiples of λ.] Let $z = x + iy$. Now the number of $\{z_n\}$ in each of the rectangles $p\lambda \leqslant x \leqslant (p+1)\lambda$, $|y| \leqslant r$, $p = 0, 1, \ldots$ is finite. Hence the number of points of E in C_r arising from those $\{z_n\}$ for which $R(z_n) \geqslant 0$ is finite. By hypothesis we can find p so great that there are no points of $\{z_n\}$ with $R(z_n) < -p\lambda$ and $|I(z_n)| \leqslant r$. Hence the points of E in C_r arising from those $\{z_n\}$ with $R(z_n) < 0$ is also finite. Hence the number of points of E in C_r is finite.

THE CONDITION IS NECESSARY. By Theorem 10, the set E cannot have finite limit points and E will be the zeros of the solution $f(z) \not\equiv 0$. Suppose now that the condition of the theorem is not satisfied. We shall show that this leads to a contradiction. We consider three cases:—

CASE (I). The $\{I(z_\mu)\}$ with $R(z_n) < 0$ contains an infinity of distinct points having a limit point α. Then the rectangle $0 \leqslant x \leqslant \lambda$, $|y - \alpha| \leqslant 1$ will contain an infinity of distinct points of E which is impossible.

CASE (II). $I(z_n) = \theta$ for all but a finite number of values of n for which $R(z_n) < 0$. We may suppose $\theta = 0$ so that $z_n = R(z_n) < 0$. If the set $\{z_n < 0\}$ contains a sub-sequence such that no two terms of this sub-sequence differ by multiples of λ, then the segment $0 \leqslant x \leqslant \lambda, y = 0$ contains an infinity of distinct points of E which is impossible.

CASE (III). Suppose now that under case (ii) $(\theta = 0)$ the zeros $z_n < 0$ are such that all but a finite number of

them differ from one another by multiples of λ. If θ_1 and θ_2 are zeros of $g(z)$ of orders p and q and $\theta_1=\theta_2+m\lambda$ for some positive integral m, then $\theta_1+\lambda$ is a zero of $f(z)$ of order at least $p+q$ as is seen from (1). Using this result we see that $z_i+\lambda$ (which z_i is the first zero of $g(z)$ with $z_n<0$) is zero of infinite order for $f(z)$ in the case under consideration, that is, E contains one particular number repeated infinitely often. This also is impossible. Hence the condition is necessary. So the theorem is proved.

6.2. REMARK. It follows from Theorem 11, that the solution $f(z)$ of (1) envisaged in the proof of Theorem 10 cannot have zeros of the form $z_0-m\lambda$ for all $m=1, 2,\ldots$ since these will give rise to zeros of $g(z)$ of the form $z_i-m\lambda$ for an infinity of positive integral values of m, where z_i is a zero of $g(z)$; this is not possible by Theorem 11. This solution $f(z)$ of (1) when $g(z)$ is a function with zeros will be called a *canonical solution* and is unique except for a factor without zeros.

6.3. THEOREM 12. *Let $g(z)$ have zeros and let $f(z)$ be a canonical solution of* (1). *Then any other solution of* (1) *is of the form $f(z)\phi(z)$, where $\phi(z)$ is an integral function with period λ.*

PROOF. Let $F(z)$ be any solution of (1) and $\phi(z)=F(z)/f(z)$. Then $\phi(z)$ is a meromorphic function with period λ. Since $F(z)=f(z)\,\phi(z)$ is an integral function, it follows that the poles, if any, of $\phi(z)$ must be zeros of $f(z)$. If z_0 is a pole of $\phi(z)$ then $z_0\pm m\lambda$, $m=1, 2, 3,\ldots$ must also be poles. Hence $z_0-m\lambda$ must be zeros of $f(z)$ for all $m=1, 2,\ldots$ which is impossible as explained in §6.2. Hence $\phi(z)$ cannot have any poles so that it is an integral function.

6.4. As explained in §5.1, if $g(z)$ in (1) has no zeros and is of order ρ, there is a solution $f(z)$ of order $\rho+1$ for (1). But this is not in general true when $g(z)$ has zeros. For instance if $g(z)$ is the canonical product

with the zeros $\{-n+i \log n\}$, $n=1, 2, \ldots$, then $g(z)$ is of order one. But the zeros of the canonical solution (when $\lambda=1$) will be $\{m-n+i \log n\}$, $m, n=1, 2, \ldots$. But $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 1/|m-n+i \log n|^\rho$ does not converge for any finite ρ since

$$\sum_{n=1}^{\infty} \frac{1}{|m-n+i \log n|^\rho} \geqslant \frac{1}{(\log m)^\rho}, \; m \geqslant 2.$$

But with an additional restriction on the situation of the zeros of $g(z)$ we can prove that (1) has a solution of order $\rho+1$ if $g(z)$ is of order ρ.

THEOREM 13. *Let $\{z_n\}$, $z_n=r_n e^{i\theta_n}$, be the zeros of $g(z)$ which is of order ρ. Let $\delta>0$ exist such that $|\theta_n| \leqslant \pi-\delta$. Then* (1) *has a solution of order* $1+\rho$.

PROOF. Under the conditions of the theorem, $\{z_n\}$ satisfies the conditions of Theorem 11. In view of the results in §§5 and 5.1, it is enough to prove that if the exponent of convergence of $\{z_n\}$ be ρ, then that of $\{z_n+m\lambda\}$, $m=1, 2, \ldots$ is $\rho+1$. Now let $\mu>1$. Then

$$0 \geqslant \sum_{m=1}^{\infty} \frac{1}{|z_n+m\lambda|^\mu} - \int_1^\infty \frac{dt}{|z_n+\lambda t|^\mu} \leqslant \frac{1}{|z_n+\lambda|^\mu}$$

by the usual integral test. Putting $\lambda t=r_n v$ (we suppose $r_n \neq 0$) we get

$$\int_1^\infty \frac{dt}{|z_n+\lambda t|^\mu} = \frac{1}{\lambda r_n^{\mu-1}} \int_{\lambda/r_n}^\infty \frac{dv}{|v+e^{i\theta_n}|^\mu}.$$

Since $|\theta_n| \leqslant \pi-\delta$, it follows that $\sqrt{(v^2+1-2v \cos \delta)} \leqslant |v+e^{i\theta_n}| \leqslant v+1$. This shows that there exist fixed positive numbers A and B such that

$$\frac{A}{r_n^{\mu-1}} \leqslant \sum_{m=1}^{\infty} \frac{1}{|z_n+m\lambda|^\mu} \leqslant \frac{B}{r_n^{\mu-1}}$$

for all n after a certain stage. Hence the exponent of

convergence of $\{z_n+m\lambda\}$ is $\rho+1$ if that of $\{z_n\}$ be ρ. As the example cited above shows, if the condition on $\{z_n\}$ given in the theorem be *violated*, there *may be no solution of finite order* at all for (1).

6.5. SOME SPECIAL CASES.

(1) $g(z)=a\neq 0$. Here $f(z)=e^{z/\lambda.\log a}$ is a solution of (1) $(\log\lambda+\pi i)\, z/\lambda$.

(2) $g(z)=z$. Here $f(z, 0)=e^{z/\lambda.(\log\lambda+\pi i)}$ is a canonical solution of (1).

(3) $g(z)=1-\frac{z}{a}$. Here

$$f(z)=f(z, a)=e^{z/\lambda\,(\log\lambda/a)}.\ \frac{1}{\Gamma\left(\frac{a-z}{\lambda}\right)}$$

is a canonical solution of (1).

(4) $g(z)=a\, z^h \prod_{i=1}^{n}\left(1-\frac{z}{a_i}\right)$. Here

$$f(z)=e^{z/\lambda.\log a}\{f(z, 0)\}^h \prod_{i=1}^{n} f(z, a_i)$$

is a canonical solution.

(5) As an illustration of Theorems 11 and 13, let $\lambda=1$ and $g(z)=\frac{1}{\Gamma(-z)}$. Let $\phi(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)^n e^{z+z^2/2n}$. Then we can find a and b so that

$$f(z)\doteq e^{az^2+bz}\phi(z)$$

is a canonical solution of (1) and this is the only canonical solution of finite order (here $\rho=1$ and $\rho+1=2$) but for a constant factor. A little calculation shows that $a=\frac{1}{2}(\gamma-1), b=-\frac{1}{2}(\gamma-1)-i\pi-\log\phi'(1)$, where γ is Euler's constant. On the other hand if $g(z)=\frac{1}{\Gamma(z)}$, the equation (1) has no other solution than $f(z)\equiv 0$.

THE RING OF ENDOMORPHISMS FOR WHICH EVERY SUBGROUP OF AN ABELIAN GROUP IS INVARIANT.

BY

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[Received 14 October, 1946].

Given an additive Abelian group A, the invariance of a subgroup B of A for an endomorphism α of A is a dyadic relation $\alpha\rho B$ between endomorphisms and subgroups of A. From every dyadic relation, two reciprocal lattices of "closed" sets are derived*. In the present case, the closed sets of endomorphisms are rings and the closed sets of subgroups are all the possible systems of admissible subgroups of A. To the largest group-theory which A affords, there corresponds the ring of those endomorphisms for which every subgroup of A is invariant. This ring $R(A)$ contains obviously all the endomorphisms $\{m\}$ which map $a \to ma$, but it may also contain other endomorphisms. It will be shown that $R(A)$ depends only on the orders of the elements which occur in A. If there is in A an element of order 0, then $R(A)$ is isomorphic to the ring of the integral numbers; otherwise it is homomorphic to the direct sum of rings of p-adic numbers, p running over all the prime numbers.

Obviously an endomorphism α belongs to $R(A)$ if and only if every cyclic subgroup remains invariant for α. Hence α maps $a \to ma$, where m is determined mod *order a* only, i.e. to every a, a class of residues $M(a)$ mod *order a* is coordinated such that α maps a on ma if and only if $m \in M(a)$. If *order* $a = 0$ (i.e. if a generates an infinite cyclic group), then $M(a)$ consists of a single number. The

* Garrett Birkhoff: 'Lattice theory', *Amer. Math. Soc. Coll.* XXV, p. 24-26.

lowest common multiple of the orders of the elements of a group G is denoted by $exp\ G$; thus $exp\ G = 0$ when there exists in G an element of order 0, or when there is no upper bound for the orders. Let a, b be two arbitrary elements of the given Abelian group A, and B the subgroup generated by them, then B admits a basis c, d and $\alpha \in R(A)$ maps $c \to kc$, $d \to md$, $c+d \to n(c+d)$. Hence $nc = kc$, $nd = md$, and therefore B is mapped by α in the same way as by $\{n\}$. The corresponding holds for every subgroup generated by a finite number of elements:

$M(a_1), M(a_2), \ldots, M(a_r)$ have a common element.

Hence if *order* $(a_1) = 0$, then α is equivalent with some $\{n\}$ and $R(A)$ is isomorphic to the ring of the integral numbers. When A is infinite, $R(A)$ is homomorphic to the ring of the endomorphisms $\{n\}$ and isomorphic to the ring of the classes of residues mod $exp\ A$. Further *order* $a =$ *order* b implies $M(a) = M(b)$ and when *order* a is a factor of *order* b, then $M(a) \supseteq M(b)$.

Suppose A is infinite and every element is of order $\neq 0$, then A is the direct sum of its primary components

$$A = P_1 + P_2 + \ldots$$

Consider one primary component P belonging to the prime number p. If *order* $a = p^r$, *order* $b = p^{r+s}$, then the elements of $M(a)$ are congruent $c_r = d_0 + d_1 p + \ldots + d_{r-1} p^{r-1}$ (mod p^r) and those of $M(b)$ are congruent $c_r + d_r p^r + \ldots + d_{r+s-1} p^{r+s-1}$ (mod p^{r+s}). Hence we obtain all the sets M from initial sections of one p-adic number. On the other hand, let such a p-adic number be given and c_r its initial sections for $r = 1, 2, \ldots$, then the mapping α: $a \to c_r a$ (for $p^r =$ *order* a) leaves every cyclic group invariant and as $c_r a = c_{r+s} a$, $\alpha(a+b) = \alpha a + \alpha b$. Hence α is an endomorphism and therefore $\alpha \in R(P)$. This ring is therefore homomorphic to the ring of the p-adic numbers. If $exp\ P = 0$, the homomorphism is an isomorphism; if

exp $P = p^m$, then $R(P)$ is isomorphic to the ring of the classes of residues mod p^m.

Let A be a direct sum $A = S + T$, where the orders of the elements of S are relatively prime to the orders of the elements of T and therefore there is no element of order 0 in A. Then every $\alpha \in R(A)$ generates in S a $\beta \in R(S)$ and in T a $\gamma \in R(T)$. We shall show that conversely when β and γ are given, the mapping α:

$$\alpha(s+t) = \beta s + \gamma t \text{ (when } s \in S, t \in T)$$

belongs to $R(A)$ and therefore $R(A)$ is a direct sum

$$R(A) = R(S) + R(T).$$

Given two arbitrary elements of A: $a' = s' + t'$ and $a'' = s'' + t''$, then s', s'' generate a finite subgroup S_1 of S and similarly t', t'' generate $T_1 \subseteq T$. *Exp* $S_1 = \sigma$, *exp* $S_2 = \tau$, $(\sigma, \tau) = 1$. In S_1, β is equivalent to some $\{m\}$ and in T_1, γ is equivalent to $\{n\}$ both groups being finite. Then there exists a number $w \equiv m \pmod{\sigma} \equiv n \pmod{\tau}$ and α is equivalent to $\{w\}$ in $S_1 + T_1$. Hence $\alpha(a' + a'')$ and the cyclic group generated by a' is invariant. Thus $\alpha \in R(A) = R(S) + R(T)$. By mathematical induction we obtain

$$R(A) = R(P_1) + R(P_2) + \ldots.$$

To obtain $R(A)$, we have therefore to know only the lattice of the orders of the elements of A (the lattice operations being h.c.f. and l.c.m.). If the lattice has a supremum, e.g. zero, then it is $=$ *exp* A and $R(A)$ is isomorphic to the ring of the classes of residues mod *exp* A. If there is no supremum, one has to consider the sublattices of the prime number powers that occur. If the sublattice belonging to the prime number p is of finite dimension n, then $R(P)$ is isomorphic to the ring of the residue classes mod p^n, otherwise to the ring of the p-adic numbers.

A NOTE ON THE PARITY OF $p(n)$

BY

HANSRAJ GUPTA.

[Received 26 August, 1946].

In 1920, Mac Mahon* gave a method for determining the parity of $p(n)$—the number of unrestricted partitions of n. He made use of Jacobi's Elliptic products. His results are here obtained in a simpler and more direct way.

We have

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x\left\{\prod_{r=1}^{\infty}(1-x^r)\right\}^{24},$$

so that

$$\left\{\prod_{r=1}^{\infty}(1-x^r)^{32}\right\}^{-1}. \quad \sum_{n=1}^{\infty}\tau(n)\, x^{n-1} = \left\{\prod_{r=1}^{\infty}(1-x^r)^8\right\}^{-1}$$

Since

$$(1-x^r)^8 \equiv 1-x^{8r} \pmod 2, \qquad (1)$$

$$(1-x^r)^{32} \equiv 1-x^{32r} \pmod 2, \qquad (2)$$

and†

$\tau(n)$ is odd if and only if n is an odd square, (3)

we get

$$\left\{\prod_{r=1}^{\infty}(1-x^{32r})\right\}^{-1}. \quad \sum_{i=0}^{\infty} x^{(2i+1)^2-1} \equiv \left\{\prod_{r=1}^{\infty}(1-x^{8r})\right\}^{-1} \pmod 2.$$

i.e.

$$\sum_{m=0}^{\infty} p(m)\, x^{32m}. \quad \sum_{i=0}^{\infty} x^{4i(i+1)} \equiv \sum_{n=0}^{\infty} p(n)\, x^{8n} \pmod 2.$$

Comparing the coefficients of like powers of x, we have

$$p(n) \equiv \sum_t p(t) \pmod 2, \qquad (4)$$

* MacMahon, *Proc. Cambridge Phil. Soc.* 20 (1921), 281-3.

† Hansraj Gupta, *Proc. Benares Math. Soc.* 5 (1943) 17-22.

where t runs through the positive integral values given by the relation

$$t = \frac{2n - i(i+1)}{8}, \; i \geqslant 0.$$

Evidently, we need give to i only those values for which

$$i(i+1) \equiv 2n \pmod 8. \qquad (5)$$

These results are in agreement with those of MacMahon.

ON THE THE ABERRANCY CURVE.*

BY

B. K. SEN GUPTA, *Srikhanda, (Bengal).*

4. The origin is an undulation on the curve $y = px + qx^2/2! + \ldots$ if $q = r = 0$. The point on the aberrancy curve corresponding to the origin is $x = -3qr/P$, $y = 3q(3q^2 - pr)/P$, where $P = 3qs - 5r^2$. When $q = r = 0$, Lt. $dP/d\theta = 3qt - 7rs = 0$ and Lt. $d^2P/d\theta^2 = -7s^2$. The limiting values of x and y at the origin are respectively given by $-3r$. Lt. $q/P = -3r$. Lt. $r/dp/d\theta = -3r$. Lt. $s/d^2p/d\theta^2 = 3r/7s = 0$, and $3(3q^2 - pr)$. Lt. $q/P = 0$, i.e. $x = 0, y = 0$ on the aberrancy curve. So the origin also lies on the aberrancy curve.

For the aberrancy curve dy/dx at the origin is $(pr - 3q^2)/r = p - 3q^2/r = p$ since Lt. $q^2/r = 0$. Hence dy/dx is the same as that of the given curve; so at the point of undulation, the curve and the aberrancy curve have got the same tangent.

d^2y/dx^2 of the aberrancy curve is $q.\ P^3/r^3.\ T = q.\ (r^3/T).\ (P/r^2)^3$, where T (the Mongian) $= 40r^3 - 45qrs + 9q^2t$; Lt. $dT/d\theta = 0$ at the origin; Lt. $T/r^3 = 35/2$ or Lt. $r^3/T = 2/35$; Lt. $P/r^2 = -7/2$; hence $d^2y/dx^2 = q.\lambda$ (constant) $= 0$.

*This section was inadvertently omitted from the paper, "On the Aberrancy curve", J.I.M.S. Vol. IX, 77-9. Ed.

d^3y/dx^3 of the aberrancy curve $= P^4.\ [P(r^2T-3qsT-qr.\ dT/d\theta)+3qrT.\ dp/d\theta]/r^5.\ T^3=(P/r^2)^5.\ (r^3/T)^2(r-3qs/r-q/T.\ dT/d\theta)+3(P/r^2)^4.\ (r^3/T)^2.\ q/r^2.dp/d\theta.$ Now Lt. $3qs/r=3s.$ Lt $q/r=0$; Lt. $q/T.\ dT/d\theta=dT/d\theta.$ Lt. $q/T=dT/d\theta.\ r/dT/d\theta=0$; Lt. $q/r^2.\ dp/d\theta=dp/d\theta.$ Lt. $q/r^2=dp/d\theta.\ 1/2s=0.$ Hence $d^3y/dx^3=0.$

From the above statements we come to the following

THEOREM. *The undulations on a given curve are also undulations on the aberrancy curve, and the tangents at them are the same for both.*

INDIAN MATHEMATICAL SOCIETY.

THE NARASINGA RAO Medal for Mathematical Research.

NEXT PRIZE PROBLEM.

To make a contribution to the theory of plane projective geometries particularly the following types of non-desarguesian geometries:

(*i*) The geometries where a single (p, L) Desargues theorem holds, where L is a line passing through the point p. [The (p, L) Desargues theorem is that which states that if the three joins of the corresponding vertices of two triangles concur at p, and if two pairs of their corresponding sides intersect on L, then the third pair also intersect on L.]

(*ii*) The geometries in which there is a transitive group of translations.

(*iii*) The geometries in which the theorem of the complete quadrangle (that is, the existence of a unique harmonic conjugate C' of C with respect to AB for every triad ABC of collinear points) holds.

(*iv*) The required "contribution" is to bear on the following questions:—

1. The distribution of points p and lines L passing through p, such that the (p, L) Desargues theorem is true, the distribution of collinear triads of points A, B, C, such that C has a unique harmonic conjugate with respect to AB.

2. The distribution of quadrangles with three collinear diagonal points.

3. The projective group of the plane and the projective groups of its straight lines.

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According to the conditions of award of the Narasinga Rao Gold Medal for Mathematical Research, any person of Indian domicile may compete for the prize if he is a member of the Society both at the time of competing and at the time of award. The medal shall be awarded for the best solution or contribution towards the solution of a specified problem in mathematics which shall be announced sufficiently in advance of the date fixed for the submission of the theses.

The solutions of the prize problem are to be sent to the President of the Society at the time, and the last date for the submission of the theses is the 1st July, 1948.

A. NARASINGA RAO,
Editor of *The Mathematics Student.*

A WEAK HOMEOMORPHISM BETWEEN TOPOLOGICAL SPACES AND A CHARACTERIZATION OF COMPLETELY REGULAR SPACES.*

BY

V. S. KRISHNAN, *Presidency College, Madras.*

[Received 17 January, 1947.]

INTRODUCTION.

The completely regular spaces form a link between the abstract topological spaces and the Euclidean or metrical spaces. Tychonoff defines such spaces as those which admit a characteristic functional for each pair of subsets, of which one consists of a single point and the other is a closed set not containing the point. He also assumes that the space is a Hausdorff space. A. Weil has given another characterization of these spaces by considering the possibility of assigning suffixed systems of neighbourhoods for the points, which satisfy the usual neighbourhood axioms uniformly with respect to the points. The fundamental property of this space, established by Tychonoff, is that such a space can always be imbedded homeomorphically in a tore [or topological product of a family of spaces each homeomorphic with the space τ of real numbers in the unit segment (0, 1)].

We consider here a way of characterizing the topology of a completely regular space from among all the other topologies possible on the same basic set of points. These topologies on a fixed set of points are easily shown to form a complete lattice when partially ordered by the relation of one topology being *weaker than* another. We now show that in this lattice the topologies which lead to completely regular spaces can be characterized as lattice

* An abstract of this paper was read at the Fourteenth Conference of the Indian Mathematical Society held at Delhi in December, 1945.

products of topologies which give regular, strongly separable (not necessarily T_1) spaces; they can also be characterized as lattice products of topologies defined by quasi-metrics [i.e., by distance functions ρ satisfying, for all x, y, z: $\rho(x,y) \geqslant 0$, $\rho(x,y)=\rho(y, x)$, $\rho(x, z) \leqslant \rho(x,y) + \rho(y, z)$]. These topologies need not even be T_0. That we cannot restrict ourselves to T_0 topologies is seen from the fact that we can have Hausdorff bicompact spaces which are not strongly separable and also not metrizable*; and these spaces are completely regular but cannot possibly have their topologies expressed as lattice products of stronger T_0 topologies defined by quasi-metrics or of stronger T_0, regular strongly separable topologies, as any stronger Hausdorff topology is necessarily identical with the original topology for a Hausdorff bicompact space.†

The above characterizations of completely regular spaces are valid even when the spaces are not T_0. Similarly Tychonoff's result on imbeddability of a T_0, completely regular space in a tore is generalized here for the non-T_0 cases by replacing the tore by a topological product of spaces all of which are regular and strongly separable or all quasi-metrizable. These characterizations are dealt with in the last section of this paper.

To facilitate the study of general T-spaces and to characterize the special place of T_0 spaces among them, we introduce in the first section, a form of homeomorphism between spaces which is weaker than the usual topological homeomorphism defined in terms of one-one continuous transformations. For the study of these we have to consider notions of continuity among *many-many transformation* between spaces. For, while the inverse of a many-

* A tore of non-enumerable dimension is an example of such a space.

† For a proof of this see (8)—p. 103, §31.7. [(8) refers to the numbered reference in the bibliography at the end].

one transformation is not many-one, in the case of many-many transformations, as in the case of one-one transformations, the original transformation and its inverse appear on an equal footing. So we take up in some detail the study of the notions of continuity and strong continuity (new to the best of my knowledge) of these many-many transformations. This leads to the fruitful notion of the weak-homeomorphism.

In the second section we consider some topological properties which are invariant under weak homeomorphisms also, and show that some of them are also cogredient [i.e. true of all subspaces of any space for which they hold]. In the next section we prove a general theorem relating to the imbeddability of a space in a topological product of special spaces with the possibility of expressing its topology as a lattice product of special topologies. The properties which serve to characterize these special spaces and special topologies have to be invariant under weak homeomorphisms and also cogredient. This theorem in combination with the results of the second section lead to the characterizations of the completely regular space mentioned earlier.

I wish to take this opportunity to express my grateful thanks to Dr. R. Vaidyanathaswamy, for his suggestions and criticisms in the preparation of this paper.

1. Many-many transformations between topological spaces.

1.1. Given an abstract set R of arbitrary elements $(x, y, z, \ldots)$, the family $B(R)$ of subsets of R form a complete Boolean algebra* with the *set-inclusion* relation $\subset$ as ordering relation; the lattice *sum* and *product* of any family x_λ of the subsets are respectively their *set-union* $\cup x_\lambda$ and their *set-intersection* $\cap x_\lambda$. The units 0, 1 of $B(R)$ are the null set O and the whole set R; and the Boolean complement of an x, $\subset R$, is its set-complement $C(x)$.

* For definitions from lattice theory see (2).

Given two abstract set R, R', a transformation f of R on R' is a mode of associating to each element x in R a non-null subset $f(x)$ of R', in such a way that $R' = \cup f(x)$, $x \epsilon R$.

Such a transformation f can be extended into a transformation f of $B(R)$ on a subset of $B(R')$, by setting $f(X) = \cup f(x)$, $x \epsilon X$ if X is a non-null subset of R and $f(O) = O'$, O' denoting the null-set viewed as a subset of R'.

Given the transformation f of R on R', a transformation f^{-1} of R' on R, called *the inverse* of f, can be defined by setting $f^{-1}(x') = [\exists x : x' \epsilon f(x)]$,* for each x' in R'. Clearly f is the inverse of f^{-1}.

For any transformation f of R on R', and its inverse f^{-1}, these being extended to $B(R)$, $B(R')$ as above, we have the following consequences from the above; [here $X, Y, X_\lambda, \ldots$ denote any subset of R, and $X', Y', X_\lambda' \ldots$ any subset of R']:

(*a*) if $X \subset Y$ then $f(X) \subset f(Y)$; if $X' \subset Y'$ then $f^{-1}(X') \subset f^{-1}(Y')$.

(*b*) $f(O) = O', f(R) = R'$; $f^{-1}(O') = O, f^{-1}(R') = R$.

(*c*) $X \subset f^{-1}f(X) \{ = f^{-1}[f(X)] \}$; $X' \subset ff^{-1}(X')$ $\{ = f[f^{-1}(X')] \}$.

(*d*) $X \cap f^{-1}(X') = O$ if, and only if, $f(X) \cap X' = O'$.

(*e*) $f[\cup (X_\lambda)] = \cup [f(X_\lambda)]$; $f^{-1}[\cup (X_\lambda')] = \cup [f^{-1}(X_\lambda')]$.

(*f*) $X = f^{-1}f(X)$ if, and only if, $f[C(X)] = C[f(X)]$; $X' = ff^{-1}(X')$ if, and only if, $f^{-1}[C(X')] = C[f^{-1}(X')]$.

(*g*) If each $X_\lambda = f^{-1}f(X_\lambda)$, then $f[\cap (X_\lambda)] = \cap [f(X_\lambda)]$; if each $X_\lambda' = ff^{-1}(X_\lambda')$, then $f^{-1}[\cap (X_\lambda')] = \cap [f^{-1}(X_\lambda')]$.

All except (*f*) and (*g*) can be proved without any difficulty. Since f, f^{-1} enter symmetrically in their relations with R, R' we need prove only the first parts in (*f*), (*g*) also. Since $X \cup C(X) =$

* Read: "the set of x's for which $x' \epsilon f(x)$", the notation is as in (5).

R, it follows by (*e*) that $f[C(X)] \supset C[f(X)]$. With (*c*), this leaves us to prove only that $X \supset f^{-1}f(X)$ if, and only if, $f[C(X)] \subset C[f(X)]$, and this can be deduced from (*d*) by replacing X, X' in it by $C(X)$, $f(X)$ respectively. To prove the first part of (*g*) now, by (*a*) we have $f[\cap (X_\lambda)] \subset \cap [f(X_\lambda)]$ always. If each $X_\lambda = f^{-1}f(X_\lambda)$, and an element x' of R' is in $\cap [f(X_\lambda)]$, $x' \varepsilon$ each $f(X_\lambda)$ and so $f^{-1}(x') \subset$ each $f^{-1}f(X_\lambda) = X_\lambda$ and so $\subset \cap X_\lambda$; hence $x' \varepsilon ff^{-1}(x') \subset f[\cap (X_\lambda)]$, proving the reverse inclusion $\cap [f(X_\lambda)] \subset f[\cap (X_\lambda)]$ also.

When f is a many-one transformation of R on R', $f(x)$ contains just one element x' for each x in R, and we write $f(x) = x'$ also. Clearly $X' = ff^{-1}(X')$ for each subset X' of R' now. Hence by (*e*), (*f*) and (*g*) f^{-1} would now carry over unions, intersections and complements of sets into unions, intersections and complements of the transforms.

When f and f^{-1} are both many-one, then f, f^{-1} are also one-one,* and R, R' are sets of the same cardinal number.

1.2. We next consider two topological spaces R, R'. The topology of R is specified by the Σ-ring, $\Gamma(R)$, of the *open subsets* of R, or equally by the Δ-ring, $\Omega(R)$, of the complementary closed subsets of R. $\Gamma(R)$, $\Omega(R)$ are sub-lattices of $B(R)$, and each consists of the complements of the elements in the other. Hence the sublattice, $\Gamma\Omega(R)$, of $B(R)$ generated from their union, [and consisting of finite unions of sets of the form $G \cap F$, $G \varepsilon \Gamma(R)$, $F \varepsilon \Omega(R)$], is also a Boolean subalgebra of $B(R)$, with the same complementation. Similar remarks apply to the space R' and the sub-lattices $\Gamma(R')$, $\Omega(R')$, $\Gamma\Omega(R')$ of $B(R')$.

If f is a transformation of the space R on the space R', then we shall say that:

(1) f is an *open transformation*, if $f(G)$ is an open set of R', whenever G is an open set of R;

(2) f is a *closed transformation*, if $f(F)$ is a closed set of R', whenever F is a closed set of R;

* This is true only when f, f^{-1} are 'onto' maps as here. Refer (1) and (5) for definitions of 'into' and 'onto' maps.

(3) f is a *continuous transformation*, if $f^{-1}(G')$ is an open set of R and $f^{-1}[C(G')] = C[f^{-1}(G')]$, whenever G' is an open set of R'.

(4) f is a *strongly continuous transformation*, if it is continuous and further every open set of R is of the form $f^{-1}(G')$, where G' is some open set of R'.

It is clear that in the definitions (3), (4) we can replace 'open' sets by 'closed' sets throughout. Again when f is a many-one transformation the definitions (3), (4) reduce to the usual ones of many-one continuous or strongly continuous transformations,* since f^{-1} preserves complements.

Because of result (f) of §1.1 we can also reframe the definitions (3) and (4) as follows [and here also we may replace 'open' sets by 'closed' sets throughout]:

(3)′ f is *continuous*, if $f^{-1}(G')$ is an open set of R and $G' = ff^{-1}(G')$, for every open set G' of R'.

(4)′ f is *strongly continuous*, if $f^{-1}(G')$ is an open set of R and $G' = ff^{-1}(G')$, for every open set G' of R', and every open set G of R is of the form $f^{-1}(G')$ for some open set G' of R'.

These forms of the definitions enable us to establish the following results.

1.21. THEOREM. *If f denotes a transformation of a space R on another R', then the following statements are all logically equivalent*:

(*a*) *f is strongly continuous*;

(*b*) *f^{-1} is strongly continuous*;

(*c*) *f, f^{-1} are both continuous.*

PROOF. We shall show that (*a*) and (*c*) imply each other. Then by interchanging the roles of f, f^{-1} and of R, R' in these proofs the equivalence of (*b*) and (*c*) would follow, so that (*a*), (*b*), (*c*) would all be equivalent.

* For treatment of these many-one types refer (1), p. 65 and (8), p. 120.

If f is strongly continuous it is continuous also. Also, by (4)′, if G is any open set of R, $G=f^{-1}(G')$, where G' is an open set of R'; and so, by (4)′ again, $G'=ff^{-1}(G')=f(G)$, so that $f(G)$ is open and $G=f^{-1}f(G)$ for each open set G of R. This proves, by (3)′—since $(f^{-1})^{-1}=f$, that f^{-1} is continuous. Hence (*a*) implies (*c*).

If f, f^{-1} are continuous then, by (3)′, the continuity of f^{-1} ensures that any open set G of R satisfies $G=f^{-1}f(G)$, and $(f^{-1})^{-1}=f$ is a open transformation. So $G'=f(G)$ is open in R' and $G=f^{-1}(G')$. Hence any open set G of R is of the form $f^{-1}(G')$, where G' is an open set of R'. This shows that f is strongly continuous. Thus (*c*) implies (*a*) also; or (*a*), (*c*) are equivalent.

One other consequence of the definition (3)′ may be noted here as it will be of use hereafter.

1.22. *When f is a continuous transformation of R on R', the subsets of R of the form $f^{-1}(G')$, $G'\epsilon\Gamma(R')$, form a Σ-ring $\Gamma'(R)$ contained in $\Gamma(R)$.* So these sets may be taken as all the open sets of R under a topology stronger than the original; and this topology for R is stronger than any for R under which f is a continuous transformation of R on R'. Also only when R has this topology is f a strongly continuous transformation of R on R'.

We observe *that* $\Gamma(R)$ is a Σ-ring *since*, by (*e*), (*g*) of §1.1, f^{-1} preserves unions and intersections of the sets of $\Gamma(R')$, since each G' in $\Gamma(R')$ satisfies $G'=ff^{-1}(G')$. The other assertions above follow from the definitions (3)′ and (4)′.

Finally we characterize the continuous and strongly continuous transformations in terms of their effect on the lattices of open or closed sets. These results will be seen to be consequences of the properties of these types of transformations which have been discussed so far.

1.23. A transformation f of a space R on another R' is:

(*a*) *continuous*, if f^{-1} is an (order—) isomorphism between the Boolean algebra $\Gamma\Omega(R')$ and a Boolean subalgebra of $\Gamma\Omega(R)$ such that $f^{-1}[\Gamma(R')] \subset \Gamma(R)$ and $f^{-1}[\Omega(R')] \subset \Omega(R)$;

(*b*) *strongly continuous*, if f^{-1} is an (order—) isomorphism between $\Gamma\Omega(R')$ and $\Gamma\Omega(R)$ such that

$$f^{-1}[\Gamma(R')] = \Gamma(R) \text{ or, (and),}$$
$$f^{-1}[\Omega(R')] = \Omega(R).$$

When f is strongly continuous it can be shown that f carries over isomorphically the Boolean subalgebras of $\Gamma(R)$, $\Omega(R)$ consisting of open or closed domains into the similar subalgebras of $\Gamma(R')$, $\Omega(R')$.

2. THE WEAK HOMEOMORPHISMS AND INVARIANTS UNDER THEM.

2.1. The spaces R, R' are said to be *homeomorphic*, (in symbols $R \approx R'$), if there is a one-one strongly-continuous transformation of R on R'; such a *transformation* is also called a homeomorphism.* This relation $\approx$ between pairs of spaces is an *equivalence relation*† and is usually taken as the equality relation between spaces. [For $R \approx R$; if $R \approx R'$, $R' \approx R$; and if $R \approx R'$, $R' \approx R''$ then $R \approx R''$.]

Similarly we now define two spaces R, R' to be *weakly homeomorphic*, (in symbols $R \sim R'$), if there is a (many-many) strongly continuous transformation of R on R'; and we call such a transformation also a weak-homeomorphism. That this relation is also an equivalence relation follows from the definitions (4), (4)′ and the results of §§1.21 and 1.23. Further it is clear that $R \approx R'$ implies $R \sim R'$.

2.2. Properties of spaces invariant under homeomorphisms are usually called topological properties. If we call properties of spaces which are invariant under weak-homeomorphisms *w. h. invariant properties*, or *w. h.*

* The *relation* between R, R' as well as the *transformation* are called homeomorphisms. Compare (1).

† For a study of binary relations and their classification see (4)

invariants, these have to be in the first instance, topological properties also.

2.3. Among the postulates of separation, which are important topological properties, the T_0 postulate has a special significance in the study of weak-homeomorphism. [The T_0 postulate states: Of any two distinct points of the space, at least one is contained in an open set not containing the other].

2.31. THEOREM (*a*). *If two T_0-spaces are weakly-homeomorphic then they are also homeomorphic.*

(*b*). *Any topological space is weakly homeomorphic with a (unique) T_0-space.*

PROOF. We shall first prove the following auxiliary results:

(1) a continuous transformation of a space R on a T_0-space R' is necessarily many-one;

(2) a many-one strongly continuous transformation of a T_0-space R on a space R' is necessarily a homeomorphism;

(3) * if E is the equivalence relation defined on a topological space R by setting: xEy if, and only if, $\bar{x}=\bar{y}$ in R, then the family, R/E, of classes $\{x\}$ of mutually equivalent elements can be topologised by assigning as 'open' the subsets $[\mathfrak{F}\{x\}: x \epsilon G]$ for the various open sets G of R; when so topologised R/E is a T_0-space and ϕ is a strongly continuous many-one transformation of R on R/E, when ϕ is defined by $\phi(x) = \{x\}$ for each x in R.

To prove (1), let f be a continuous transformation of a space R on a T_0-space R'. If x', y' are elements of R' in $f(x)$ for some x in R, then if x' is in an open set G' of R', $y' \epsilon f(x) \subset f[f^{-1}(x')] \subset ff^{-1}(G') = G'$. Similarly every open set of R' containing y' also contains x'. Hence, as R' is a T_0-space, x', y' cannot be distinct elements of R'

* Compare Cech's proof in (3).

when both are in some $f(x)$, $x \epsilon R$. As $f(x)$ is non-null for every x in R, it follows that $f(x)$ contains just one element of R' for each x in R; or f is a many-one transformation.

When f is a strongly continuous transformation of a T_0-space R on a space R', then f^{-1} is a continuous transformation of R' on R. So the result (1) applied to f^{-1} proves that f^{-1} is many-one. If f is also assumed to be many-one, besides being strongly continuous, it follows that f, f^{-1} are one-one and so are homeomorphisms. This proves (2).

To prove (3) we observe that any open set G of R is a union of sets $\{x\}$ belonging to R/E; for if $x \epsilon G$ and $y \epsilon \{x\}$ so that $\bar{y} = \bar{x}$, then $x \epsilon \bar{y}$; hence G must intersect (y) or contain y. Hence the sets of R/E prescribed as 'open' are the sets of the form $[E\{x\}: x \epsilon G] = [E\{x\}: \{x\} \subset G]$. Thus these subsets of R/E form a Σ-ring and so define a topology for R/E. This is a T_0-topology; for if $\{x\}$, $\{y\}$ are distinct elements of R/E, $\bar{x} \neq \bar{y}$ so that $x \epsilon \bar{y}$ and $\bar{y} \epsilon x$ cannot both be true. So there is an open set G of R which contains one of x, y but not both. Then the open set $[\exists \{z\}: z \epsilon G]$ contains one of $\{x\}$, $\{y\}$ but not both. This proves that R/E is a T_0-space. That ϕ is a strongly continuous transformation follows from the definition of the open sets of R/E.

The results (1) and (2) proved above show that a strongly continuous transformation of one T_0-space on another is necessarily a homeomorphism. So two weakly-homeomorphic T_0-spaces are also homeomorphic, which proves the part (*a*) of the theorem. The result (3) shows that any space is weakly homeomorphic to a T_0-space, which, by (*a*), is unique to within a homeomorphism. Thus is (*b*) proved.

2.32. This theorem brings out another characterestic property of weak homeomorphisms. Two-spaces R, R'

are weakly homeomorphic if, and only if, they can be mapped by many-one strongly-continuous transformations on one and the same T_0-space.

2.4. Since a non-T_0 space is weakly homeomorphic with, but cannot be homeomorphic with, a T_0-space the above result shows that the relations $\sim$, $\approx$ are two distinct equivalence relations for topological spaces (and though $R \approx R'$ implies $R \sim R'$ the reverse implication is not true in general, unless we confine ourselves to T_0-spaces only). Also we infer from the above that the T_0 postulate is a topological property which is not w.h. invariant.

2.5. We next consider some w.h. invariant properties which become identical with familiar topological properties in the case of T_0-spaces.

2.51. THEOREM. *All the following topological properties of a space R are also w.h. invariant properties:*

(1) T_1'* : if for two points x, y of R there is an open set containing x but not y, then there is also an open set containing y but not x;

(2) T_2'* : if for two points x, y of R there is an open set containing x but not y, then there exist two disjoint open sets containing x, y one each;

(3) REGULARITY : if G is any open set of R and x any point in G, then we can find an open set H and a closed set F of R such that

$$x \in H \subset F \subset G;$$

(4) NORMALITY : if F, K are any two disjoint closed subsets of R, there exist disjoint open sets G, H of R such that $F \subset G$ and $K \subset H$;

(5) SEPARABILITY : there is an enumerable subset N of R such that $\bar{N} = R$ (or if $F \supset N$, for any closed set F of R, then $F = R$);

* For a T_0-space, these give the T_1, T_2 separation postulates. See (2).

(6) STRONG SEPARABILITY: there is an enumerable family $\{G_i\}$ of open sets of R such that every open set of R is the union of a subfamily of $\{G_i\}$;

(7) COMPLETE REGULARITY: given any point p of R and any closed set F not containing p, there is a characteristic functional $\phi = \phi(p, F)$ for the pair (p, F); i.e. a continuous (many-one)* transformation of R on a subspace of the real unit segment τ, such that $\phi(p) = 0$, and $\phi(q) = 1$ for all q in F.

(8) QUASI-METRIZABILITY: there is a qusi-metric ρ on R defining its topology; i.e., there is a real valued function $\rho(x, y)$ of ordered pairs (x, y) of elements of R such that, for all x, y, z from R, $\rho(x, y) \geqslant 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$; further the topology of R is such that a subset, G, of R is open if, and only if, it contains with each point x in it a 'sphere', $U_r(x) = [Ey : \rho(x, y) < r]$, for some positive number r.

PROOF. When f is a strongly continuous transformation of R on R', then f, f^{-1} carry the lattices $\Gamma(R)$, $\Gamma(R')$ and also the lattices $\Omega(R)$, $\Omega(R')$ isomorphically into each other. Also given a subset X of R we can find a subset X' of R' which has the same cardinal number (i.e. there is a one-one transformation of X on X') and $X \subset f^{-1}(X')$, $X' \subset f(X)$. [We have only to let X' be a set of elements x' chosen one from each of the sets $f(x)$, $x \in X$]. Hence $X \subset G$ or $X \subset F$ for an open set G or closed set F of R if, and only if, $X' \subset G'$ or $X' \subset F'$, where $G' = f(G)$, $F' = f(F)$ [and $G = f^{-1}(G')$, $F = f^{-1}(F')$]. These remarks show that if any one of the properties (1) to (6) given above is true of R' then the same is true for R. So all of them are w.h. invariant properties.

If R' is completely regular or quasi-metrizable we shall now show that the same is true of R, so that these

* Since τ, and any subspace of τ are T_0-paces, the continuous transformation ϕ must be many-one, by result (1) of §2.31.

two properties are also w.h. invariant. So first let R' be completely regular. If p is any point of R and F a closed set of R not containing p, then, as above taking $X = (p)$ and $X' = (p')$ where $p' \epsilon f(p)$, as (p) is not a subset of F, (p') is not a subset of $F' = f(F)$ and F' is a closed set of R'. Hence, as R' is completely regular, there is a characteristic functional ϕ' for the pair (p', F'). Hence $\phi = \phi' f$ is a continuous transformation of R on a subspace τ' of τ such that $(0) = \phi'(p') \subset \phi' f(p)$, and $(1) = \phi'(q') \subset \phi' f(q)$ if $q' \epsilon f(q)$ and $q' \epsilon F'$. As τ' is a T_0-space, and $\phi' f$ continuous, it must be many-one. Hence $\phi(p) = \phi' f(p) = 0$, and $\phi(q) = \phi' f(q) = 1$ if ($q \epsilon f^{-1}(q')$ and $q' \epsilon F'$, or) $q \epsilon F$. Thus ϕ is a characterestic functional for the pair (p, F) and R is a completely regular space also. If we next assume R' is quasi-metrizable, and ρ' a quasi-metric on R' defining its topology, then for any two elements x, y of R we can write $\rho(x, y) = \rho'(x', y')$ where $x' \epsilon f(x)$ and $y' \epsilon f(y)$. This gives a unique value for $\rho(x, y)$ for any choice of x', y' from $f(x), f(y)$.

[For let ξ, η be from $f(x)$, $f(y)$; then since ρ' is a quasi-metric defining the topology of R', every $U_r(x')$ is open in R', so that $ff^{-1}[U_r(x')] = U_r(x')$; and $\xi \epsilon f(x) \subset ff^{-1}(x') \subset ff^{-1}[U_r(x')] = U_r(x')$. Therefore $\rho'(x', \xi) < r$; since this is true for every positive number r, and $\rho'(x', \xi) \geqslant 0$, it follows that $\rho'(x', \xi) = 0$. Similarly $\rho'(y', \eta) = 0$; and so

$$\rho'(x', y') \leqslant \rho'(x', \xi) + \rho'(\xi, y') = \rho'(\xi, y')$$
$$\leqslant \rho'(\xi, \eta) + \rho'(\eta, y') = \rho'(\xi, \eta)$$

or

$$\rho'(x', y') \leqslant \rho'(\xi, \eta).$$

Similarly $\rho'(\xi, \eta) \leqslant \rho'(x', y')$ or $\rho'(x', y')$ is the same for every choice of x' from $f(x)$ and y' from $f(y)$.]

This definition of $\rho(x, y)$ gives a quasi-metric for R; to show that the topology of R is that defined by ρ we observe that any open set G of R is of the form $f^{-1}(G')$ where G' is an open set of R'; and G' is a union of 'spheres' $U_r(x')$, $x' \epsilon G'$. As these spheres are also open sets of R'

$$G = f^{-1}(G') = f^{-1} \cup [U_{r_i}(x_i'), x_i' \epsilon G']$$

$$= \cup [f^{-1} U_{r_i}(x_i')], \; x_i' \in G'$$
$$= \cup V_{r_i}(x_i), \; x_i \in G,$$

where $V_r(x)$ is the sphere in R defined by

$$V_r(x) = [Ey : \rho(x, y) < r].$$

Hence the spheres $V_r(x)$ are open in R and all open sets of R are unions of these spheres. Hence R is quasi-metrizable, and ρ is a quasi-metric defining the topolcgy of R.

Thus the theorem is completely proved.

2.52. THEOREM. *The properties T_1', T_2', regularity, complete regularity, quasi-metrizability and strong separability are cogredient properties.*

PROOF. Let R be a space and S a subspace. The open sets of S are of the form $G \cap S$, where G is any open set of R; and $F \cap S$, for closed sets ,F, of R give the closed sets of S.

If now an open set $(G \cap S)$ of S contains x but not y, where x, y are elements of S, then $x \in G$, y is not in G. Hence if R is T_1' there is a open set H of R such that $y \in H$, x is not in H; and then $H \cap S$ is open in S, $y \in H \cap S$ and x is not in $H \cap S$. So S is T_1' if R is T_1'. If R is T_2', then $x \in G \cap S$, y is not in $G \cap S$ imply $x \in G$, y is not in G and so disjoint open sets H, J of R exist such that $x \in H$, $y \in J$. Then $H \cap S$, $J \cap S$ are disjoint open sets of S such that $x \in H \cap S$, $y \in J \cap S$, so that S is also T_2'.

If x is any element of S and $G \cap S$ any open set of S containing x (or $c(G) \cap S$ is any closed set of S not containing x), then $x \in G$, x is not in $c(G)$. Hence if R is regular, an open set H and a closed set F of R exist such that $x \in H \subset F \subset G$; then $H \cap S$, $F \cap S$ are an open and closed set respectively of S such that $x \in H \cap S \subset F \cap S \subset G \cap S$, so that S is also regular. If R is completely regular so that a characterestic function ϕ exists for the pair $[x, c(G)]$, then the same function when restricted to elements of the

subset S is a continuous many-one transformation, ϕ', of S on a subspace $\tau'' = \phi(S)$ of $\tau' = \phi(R)$ [which is a subspace of τ]; for $(\phi')^{-1}[g \cap \tau''] = S \cap \phi^{-1}(g)$ for any open subset g of τ', and $S \cap \phi^{-1}(g)$ is open in S as $\phi^{-1}(g)$ is open in R. Thus ϕ' is a many-one continuous transformation of S on a subspace τ'' of τ such that $\phi'(x) = \phi(x) = 0$, and $\phi'(y) = \phi(y) = 1$ if $y \in c(G) \cap S \subset c(G)$. Hence there is a characteristic functional to the pair $[x, c(G) \cap S]$ consisting of any point x of S and any closed set $c(G) \cap S$ of S not containing x. Therefore S is completely regular when R is completely regular.

If R is quasi-metrizable and ρ is a quasi-metric defining the topology of R, then ρ is also a quasi-metric for S ($\subset R$); and any open set of S is of the form

$$
\begin{aligned}
S \cap G &= S \cap [\cup \{ U_{r_i}(x_i) \}, x_i \in G],\\
&= \cup [S \cap \{ U_{r_i}(x_i) \}], x_i \in G,\\
&= U\, V_{r_i}(x_i),\ x_i \in G \cap S,
\end{aligned}
$$

where $V_{r_i}(x)$ for any x in $S = [Ey : \rho(x, y) < r, y \in S]$. Hence the topology of $S \cap G$ is defined by the same quasi-metric ρ applied to pairs of its elements.

Finally if R is strongly separable there is an enumerable family $\{G_i\}$ of open sets of R such that for any open set G of R we have

$$G = \cup G_j, \text{ for a subfamily } \{G_j\} \text{ of } \{G_i\};$$

so the enumerable family $\{G_i \cap S\}$ of open sets of S is such that for any open set $G \cap S$ of S we have

$$
\begin{aligned}
G \cap S &= S \cap (\cup G_j) \\
&= \cup [S \cap G_j] \text{ for a subfamily } \{S \cap G_j\} \text{ of } \{S \cap G_i\}.
\end{aligned}
$$

So S is also strongly separable. This completes the proof of the theorem.

3. Relation between imbedding a space in a topological product of special spaces and expressing its topology as a lattice product of special topologies.

3.1. We saw before, that for a space R its topology is defined by the Σ-ring of open sets. By taking different Σ-rings of subsets of R (each containing O and R) as the ring of open sets we get different topologies for R. When T, T' denote the topologies defined by two Σ-rings Γ, Γ' we say T is a *weaker topology* than T', in symbols $T < T'$, if $\Gamma \subset \Gamma'$. Then '$<$' partially orders the topologies on R and these (like the family of Σ-rings of subsets of R containing O and R) form a complete lattice under this order. The lattice sum and product of a family of topologies defined by Σ-rings (Γ_i) will be the topologies in which the open sets are respectively the sets belonging to every Γ_i, and the unions of finite intersections of sets from $(U\Gamma_i)$.

The direct product PR_λ of an indexed family of sets $\{R_\lambda\}$ consists of the indexed sets $\{x_\lambda\}$ of elements one from each R_λ. If the R_λ are topological spaces we call PR_λ the topological product of the spaces R_λ if the set PR_λ is topologised by taking as open sets the unions of finite intersections of sets of the form $H_{\mu a} = PS_\lambda$, where $S_\lambda = R_\lambda$ for $\lambda \neq \mu$, and $S_\mu = G_{\mu a} =$ an open subset of R_μ. Clearly the many-one transformations $\{\phi_\lambda\}$ of (PR_λ) on the R_λ defined by: $\phi_\lambda(\{x_\lambda\}) = x_\lambda$, are all continuous. And as we showed in the proof of w.h. invariance of completе regularity, any of these ϕ_λ restricted to a subset S of (PR_λ) will also give a continuous transformation ϕ_λ of S on $\phi_\lambda(S) = S_\lambda'$, a subspace of R_λ. We shall use the continuity of these (ϕ_λ) and (ϕ_λ') in the proof of our main theorem which follows.

3.3. THEOREM. *When R is any topological space and P is a w.h. invariant and cogredient property of spaces, the following statements are all equivalent:*

(a) R is homeomorphic with a subspace of a topological product of P-spaces;

(b) R is weakly homeomorphic with a subspace of a topological product of P-spaces;

(c) *the topology of R is the lattice product of P-topologies for R.*

[We say R is a P-space if R has the property P; and a P-topology for R is one which makes R a P-space].

PROOF. Clearly (a) implies (b).

Assuming (b), there is a strongly continuous transformation f of R on a subspace S of (PR_λ) where each R_λ is a P-space. As we remarked in the last paragraph, there is a continuous transformation ϕ_μ' of S on a subspace S_μ' of R_μ for each μ. Hence $f_\mu = \phi_\mu' f$ is a continuous transformation of R on S_μ'; as each R_μ is a P-space, so is each S_μ'. By (1.22) we can define a topology T_μ for R stronger than its original topology such that the sets of R open under T_μ are the sets of the form $f_\mu^{-1}(S_\mu' \cap G_{\mu\alpha})$, $G_{\mu\alpha}$ being an arbitrary open set of R_μ, and so $S_\mu' \cap G_{\mu\alpha}$ an arbitrary open set of S_μ'. Also when R has the topology T_μ, f_μ is a strongly continuous transformation of R on the P-space S_μ'. Hence T_μ is a P-topology for R. We shall show that the topology of R is the lattice product of the P-topologies T_μ, so that (c) follows from (b).

For $f^{-1}(G_{\mu\alpha} \cap S_\mu') = f^{-1}\phi'^{-1}_\mu (G_{\mu\alpha} \cap S_\mu')$

$$= f^{-1}[\{\phi'^{-1}_\mu (G_{\mu\alpha})\} \cap (PR_\lambda)]$$
$$= f^{-1}[H_{\mu\alpha}].$$

Now, as f is strongly continuous and as the open sets of PR_λ are unions of finite intersections of the open sets $H_{\mu\alpha}$,

any open set G of R is of the form

$G = f^{-1}$ [a union of finite intersections of sets of the form $H_{\mu\alpha}$]

= a union of finite intersections of sets of the form $f^{-1}(H_{\mu\alpha})$.

Hence the topology of R is the lattice product of the P-topologies T_μ, and (b) implies (c).

We complete the proof of the theorem by showing that (c) implies (a) also. If R has a topology, T, which is the lattice product of P-topologies T_λ, for R, then

there is a one-one transformation ϕ of (R, T)—(i.e., the set R with the topology T)—on a subspace D—(the *diagonal*) —of the topological product $P(R, T_\lambda)$ of the P-spaces (R, T_λ), defined by $\phi(x) = \{x_\lambda\} \epsilon P(R, T_\lambda)$, where each x_λ is equal to x. To see that ϕ is a homeomorphism between R and D we observe that the open sets of R are unions of finite intersections of the subsets $\{G_{\lambda\alpha}\}$ of R open under the various topologies T_λ; while any open set of D is the union of finite intersection of sets of the form $D \cap H_{\lambda\alpha}$, with $H_{\lambda\alpha} = PS_\mu$, $S_\mu = R$, for $\mu \neq \lambda$ and $S_\lambda = G_{\lambda\alpha}$ = an open set of (R, T_λ). Since $D \cap H_{\lambda\alpha} = \phi\ (G_{\lambda\alpha})$ and ϕ is one-one it follows that ϕ, ϕ^{-1} are isomorphisms between the open sets of R and D. Hence R is homeomorphic with the subspace D of the product of P-spaces $P(R, T_\lambda)$; or (c) implies (a).

4. The completely regular space and its topology.

A completely regular T_0-space is also a T_2' and a Hausdorff space. Such a space we shall call a Tychonoff space, and begin with the following characterestic property of such a space proved by Tychonoff:

4.1. A Tychonoff space is homeomorphic with a subspace of a tore τ^N i.e. of a topological product PR_λ, $\lambda \epsilon N$, of spaces R_λ each identical with the real unit segment.

From this it follows that a subspace of a Tychonoff space is a Tychonoff space, as we could also have derived from the cogredience of the properties of complete regularity and the T_0 postulate; also any topological product of Tychonoff spaces will be homeomorphic with a subspace of a topological product of tores which would also be a tore. Hence a topological product of Tychonoff space is also a Tychonoff space.

Using (4.1) we now give other characterisations of completely regular spaces.

4.2. THEOREM. *For a space R to be completely regular, each of the following conditions is necessary and sufficient:*

(1) *R is homeomorphic with a subspace of a product of regular and strongly separable spaces;*

(2) *R is weakly homeomorphic with a subspace of a product of regular and strongly separable spaces;*

(3) *the topology of R is the lattice product of regular and strongly separable topologies for R;*

(4) *R is homeomorphic with a subspace of a product of quasi-metrizable spaces;*

(5) *R is weakly homeomorphic with a subspace of a product of quasi-metrizable spaces;*

(6) *the topology of R is the lattice product of quasi-metrizable topologies for R.*

PROOF. By §§2.51 and 2.52 the properties of regularity, strong separability and quasi-metrizability are all w.h. invariant and cogredient. Hence, by §3.3, it follows that the conditions (1), (2) and (3) for a space R are logically equivalent, as also the conditions (4), (5) and (6). So the theorem will be proved if we show that R is completely regular if and only if conditions (2) and (4) are satisfied. Since the space τ is regular, strongly separable, and qusi-metrizable, it follows from §4.1 that a completely regular space satisfies the conditions (2) and (4).

Let us next assume that R is weakly homeomorphic with a subspace S of a product PS_λ of spaces S_λ each of which is (1) regular and strongly separable or (2) quasi-metrizable. Then there is a strongly continuous many-one transformation f_λ of each S_λ on a T_0-space S_λ' [by §2.31]. It is easily verified that the transformation f of PS_λ on PS_λ' defined by $f(\{x_\lambda\}) = \{f_\lambda(x_\lambda)\}$ is also a many-one strongly continuous transformation. Each S_λ' is a T_0-space weakly homeomorphic to S_λ, and so either regular and strongly separable or quasi-metrizable. In either case S_λ' must be metrizable,* and so a Tychonoff

* See (10) for a proof that a T_0, regular (and so Hausdorff) strongly separable space is metrizable.

space. But we observed earlier that PS_λ' must then be a Tychonoff space also. Hence PS_λ, which is weakly homeomorphic with PS_λ' is also completely regular. The subspace S of PS_λ is therefore completely regular and R, being weakly homeomorphic with S, is also completely regular. So the conditions (2) and (4) also imply that the space R is completely regular.

This completes the proof of the theorem.

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FURTHER APPLICATIONS OF THE IDENTICAL EQUATION TO RAMANUJAN'S SUM $C_M(N)$ AND KRONECKER'S FUNCTION $\rho(M, N)$.*

BY

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1. Introduction.

In a recent paper,(1) the author has shown that every multiplicative function $F(M, N)$ of two arguments satisfies an 'identical equation' of the form†

$$F(M, N) = F(M, 1)\, F(1, N).\quad C(M, N)$$
$$= \sum_{d/M} \sum_{\delta/N} F(M/d, 1)\, F(1, N/\delta)\, C(d, \delta), \qquad (A)$$

where $C(M, N)$ is a cardinal function, determined by and called the *cardinal component* of $F(M, N)$. It is also shown there that in the case of Ramanujan's function $C_M(N)$ the above identical equation reduces to

$$C_M(N) = E^{-1}(M)\, E(N).\ \text{princ I.}$$

Some properties of $C_M(N)$ were derived by using the identical equation. We consider here other applications of the identical equation to $C_M(N)$ and also to the function(2) $\rho(M, N)$ (defined below).

2. Properties of Ramanujan's function $C_M(N)$.

Theorem I. $\Sigma C_M(d) = \sum_{\delta/g} \mu(M/\delta)\, d(N/\delta)\, \delta$, *where the summation on the left is for all the divisors d of N, d(M) is the number of divisors of M and g is the greatest common divisor of M, N.*

For $\sum_{d/N} C_M(d) = C_M(N).\quad E_0(M)\, E(N)$

$$= E^{-1}(M)\, E(N).\ \text{princ I.}\ E_0(M)\, E(N).$$

* I am indebted to Dr. R. Vaidyanathaswamy for suggestions in the preparation of this paper.

† The notations are the same as in my paper, referred to as (1).

$$= E^{-1}(M)\ \{E(N).\quad E(N)\}.\ \text{princ I}$$
$$= \mu(M)\, d(N).\ \text{princ I}$$
$$= \sum_{\delta/M,N} \mu(M/\delta)\, d(N/\delta)\ \delta$$
$$= \sum_{\delta/g} \mu(M/\delta)\, d(N/\delta)\ \delta.$$

THEOREM 2. $\sum_{d/M}\sum_{\delta/N} C_d(\delta) = d(N/M)\ M$ *if* M/N,

and $=0$, *otherwise.*

For $\sum_{d/M}\sum_{\delta/N} C_d(\delta) = C_M(N).\ E(M)\,E(N)$

$$= E^{-1}(M)\,E(N).\ \text{princ I.}\ E(M)\,E(N)$$
$$= E_0(M)\,d(N).\ \text{princ I}$$
$$= \sum_{\delta/M,N} E_0(M/\delta)\,d(N/\delta)\ \delta$$
$$= d(N/M)\ M \text{ if } M/N \text{ and } 0 \text{ otherwise.}$$

This theorem can also be stated in the following form:—

The integral of Ramanujan's function $C_M(N)$ *with respect to both the arguments is an ordinal function with* N *as the major argument and is equal to* $d\ (N/M)\ M$ *whenever* M *divides* N.

THEOREM 3. $\sum C_{M/\delta}\ (N/\delta) = \sum_{\delta/g} \mu(M/\delta)\ \sigma(\delta)$, *where the summation on the left is over all the common divisors* δ *of* M, N *and* $\sigma(M)$ *is the sum of the divisors of* M.

For $C_{M/\delta}(N/\delta) = C_M(N).\ \text{princ } E$

$$= E^{-1}(M)\,E(N).\ \text{princ I. princ } E$$
$$= E^{-1}(M)\,\mathrm{E}(N).\ \text{princ I. } E$$
$$= \mu(M)\,E(N).\ \text{princ } \sigma$$
$$= \sum_{\delta/g} \mu(M/\delta)\,\sigma(\delta).$$

The following theorem gives a simple relation connecting $C_M(N)$ with $C_N(M)$.

THEOREM 4. *The composite of* $C_M(N)$ *and* $C_N(M)$ *is the 'principal' function equivalent to* $I(M)\,d(M)$.

We have $C_N(M) = \sum_{\delta/g} \mu(N/\delta)\ \delta$

$$= E(M)\,E^{-1}(N).\ \text{princ I.}$$

Hence

$$C_M(N).\ C_N(M)$$

$= \{E^{-1}(M)E(N).\ \text{princ I}\}.\ \{E(M)E^{-1}(N).\ \text{princ I}\}$
$= \{E^{-1}(M)E(N).\ E(M)E(N)\}.\ \{\text{princ I. princ I}\}$
$= E_0(M)\ E_0(N).\ \text{princ I.I}$
$= \text{princ } Md(M) = \text{princ I } (M)\ d(M).$

3. KRONECKER'S FUNCTION $\rho(M, N)$.

This function is defined by, $\rho(M, N) = 1$ if $g = (M, N) = 1$ and $= 0$, otherwise.

It can be easily seen that $\rho(M, N) = E_0(g)$ and is hence *multiplicative in both the arguments.* The generating series of $\rho(M, N)$ to any prime base p is given by,

$$\rho_p(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho(p^m, p^n)\ x^m y^n = \frac{1-xy}{(1-x)(1-y)},$$

on reduction.

THEOREM 5. *The cardinal component of* $\rho(M, N)$ *is the 'principal' function equivalent to* $\mu(M)$.

For, the identical equation of $\rho(M, N)$ by (A) is given by,

$$\rho(M, N) = \rho(M, 1)\ \rho(1, N).\ C(M, N)$$
$$= E(M)E(N).\ C(M, N), \qquad \text{(i)}$$

where $C(M, N)$ is the cardinal conponent.

But $\rho(M, N) = E_0\,(g) = \sum_{\delta/g} \mu(\delta)$

$$= E(M)\ E(N).\ \text{princ } \mu(M). \qquad \text{(ii)}$$

Hence, identifying (i) and (ii), we see that

$$C(M, N) = \text{princ } \mu(M).$$

NOTE. The following general theorem relating to multiplicative functions of the greatest common divisor of the arguments can be stated and easily proved :—

The cardinal component of any function $f(g)$ *of the greatest common divisor of the arguments* M, N *is the 'principal' function equiavalent to* $(f.E^{-1})\ (M)$.

For let $F(M, N) = f(g)$. Then $F(M, 1) = E(M)$ and $F(1, N) = E(N)$ and the identical equation of $F(M, N)$ is given by

$$F(M, N) = E(M)\, E(N) . C(M, N),$$

C being the cardinal component. But clearly $F(M, N)$ is a modular function with M or N as the modulus. Hence* $C(M, N)$ is an ordinal function with M or N as the major argument, i.e. $C(M, N)$ is a principal function.

Suppose $C(M, N) = \text{princ}\ \psi$.

Then $F(M, N) = E(M)\ E(N).\ \text{princ}\ \psi = \sum_{\delta/g} \psi(\delta)$.

Therefore $F(M, M) = f(M) = \sum_{\delta/M} \psi(\delta) = \psi(M).E(M)$.

Hence $\psi(M) = f(M).\ E^{-1}(M) = (f.\ E^{-1})\ (M)$ and the theorem is proved.

Theorem 6. *If $f(M)$ is the convolute of $\rho(M_1, M_2)$, then f represents the number of block factors† of M. Further,‡ $f(M) = \sum_{d/M} \mu(d) = 2^k$, where $|\mu(M)|$ is the absolute value of $\mu(M)$ and k is the number of distinct prime factors of M.*

For

$$f(M) = \Omega\, \rho(M_1, M_2) = \sum_{d/M} \rho(M/d, d).$$

From the definition of ρ,

$$\rho(M/d,\ d) = 1 \text{ if } (d,\ M/d) = 1$$
$$= 0, \text{ otherwise.}$$

Hence $f(M)$ denotes the number of block factors of M. Again the g-series of $f(M)$ to the base p is given by

$$f_p(x) = \rho_p(x, x) = (1 - x^2)/(1 - x)^2 = (1 + x)/(1 - x)$$
$$= |\mu(x)|_{(p)}\ E_{(p)}(x) = E_{2(p)}\ (x),$$

since we have seen that

$$\rho_{p}\ (x, y) = \frac{1 - xy}{(1 - x)\ (1 - y)}$$

is the generating series of $\rho(M, N)$. Therefore

$$f(M) = |\mu(M)|.\ E(M) = E_2(M)$$
$$= \sum_{d/M} |\mu(d)| = 2^k \text{ if } M = p_1^{m_1}\, p_2^{m_2} \ldots p_k^{m_k}.$$

* See (1) Theorem (3.2.2).

† A factor d of M is said to be a block factor if d is prime to M/d.

‡ See (3) and (4).

REMARK. It can be easily seen that the number of block factors of M is equal to $k_1^c + k_2^c + \ldots\ldots + k_k^c = (1+1)^k = 2^k$ and hence $f(M) = 2^k$.

We will now show how we can obtain from $\rho(M, N)$ the expression for $\phi(M)$, the number of numbers less than and prime to M. Consider

$$\sum_{r=1}^{M} \rho(M, r) = \sum_{r=1}^{M} \sum_{\delta/(r, M)} \mu(\delta).$$

The left side of the above identity is obviously equal to $\phi(M)$; while, on the right side, it can be easily seen that every divisor δ of M occurs M/δ times, for, the numbers δ, $2\delta, \ldots, \frac{M}{\delta}\delta (= M)$, and *only these*, give rise to δ once each.

Hence it follows that

$$\phi(M) = \sum_{\delta/M} \frac{M}{\delta}\mu(\delta) = M \sum_{\delta/M} \frac{\mu(\delta)}{\delta}.$$

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PARABOLIC CYLINDER FUNCTIONS WHICH ARE HANKEL-TRANSFORMS OF EACH OTHER.

BY

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1. The object of this note is to prove the following theorem:—

$$\textit{If } \psi(x) = e^{\frac{1}{4}x^2} x^{\nu+n-\frac{1}{2}} \sum_{r=0}^{m} {}_mc_r (-x)^r D_{-n-2\nu-r}(x),$$

and

$$\phi(x) = e^{\frac{1}{4}x^2} x^{\nu+m-\frac{1}{2}} \sum_{r=0}^{n} {}_nc_r (-x)^r D_{-m-2\nu-r}(x),$$

then $\psi(x)$ *and* $\phi(x)$ *are* J_ν*-transforms of each other, where m and n are positive integers and* $R(\nu-\frac{1}{2}) > 0$.

2. If we take in the operational relation[1]

$$\frac{1}{p^{\nu-1}} f(1/p) \doteq \int_0^\infty (x/t)^{\nu/2} J_\nu (2\sqrt{xt})\ h(t)\ dt,$$

where

$$f(p) \doteq h(x)$$

$$f(p) = p\frac{p^{\frac{n+1}{2}}}{(1+\sqrt{p})^{m+n+2\nu}} \doteq h(x). \qquad (2.1)$$

We have

$$\frac{p^{\frac{m+1}{2}}}{(1+\sqrt{p})^{m+n+2\nu}} \doteq \int_0^\infty (x/t)^{\nu/2} J_\nu (2\sqrt{xt})\ h(t)\ dt. \qquad (2.2)$$

In order to interpret (2.1) and (2.2), we observe that[2]

$$\sqrt{\frac{\pi}{2}} \frac{\sqrt{p}}{(1+\sqrt{p})^\lambda} \doteq e^{x/2} (2x)^{\frac{\lambda-1}{2}} D_{-\lambda} (\sqrt{2x}), R(\lambda) > 0, \qquad (2.3)$$

and as

$$\frac{p^{\frac{n+1}{2}}}{(1+\sqrt{p})^{m+n+2\nu}} = \frac{\sqrt{p}}{(1+\sqrt{p})^{m+2\nu}} \left[1 - \frac{1}{1+\sqrt{p}}\right]^n$$

$$= \sum_{r=0}^{n} {}_nc_r (-1)^r \frac{\sqrt{p}}{(1+\sqrt{p})^{m+2\nu+r}}$$

we have, on term by term interpretation with the help of (2.3)

$$\frac{p^{\frac{n+1}{2}}}{(1+\sqrt{p})^{m+n+2\nu}} \doteq h(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{x/2} (2x)^{\frac{m+2\nu-1}{2}}$$

$$\times \sum_{r=0}^{n} {}_nC_r \left(-\sqrt{2x}\right)^r D_{-m-2\nu-r}\left(\sqrt{2x}\right) \quad (2.4)$$

and

$$\frac{p^{\frac{m+1}{2}}}{(1+\sqrt{p})^{m+n+2\nu}} \doteq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{x/2} (2x)^{\frac{n+2\nu-1}{2}}$$

$$\times \sum_{r=0}^{m} {}_mC_r \left(-\sqrt{2x}\right)^r D_{-n-2\nu-r}\left(\sqrt{2x}\right). \quad (2.5)$$

Now, since by Lerch's theorem(5) every image function $f(p)$ represents operationally a unique original function $h(x)$, we obtain from (2.2) after a little simplification

$$2^{n/2} e^{x/2} x^{\frac{n+\nu-1}{2}} \sum_{r=0}^{m} {}_nC_r \left(-\sqrt{2x}\right)^r D_{-n-2\nu-r}\left(\sqrt{2x}\right)$$

$$= 2^{m/2} \int_0^{\infty} e^{t/2} t^{\frac{m+\nu-1}{2}} J_\nu\left(2\sqrt{xt}\right)$$

$$\times \sum_{r=0}^{n} {}_nC_r \left(-\sqrt{2t}\right)^r D_{-m-2\nu-r}\left(\sqrt{2t}\right) \, dt, \quad (2.6)$$

which, again, after some transformation and manipulation becomes

$$e^{\frac{1}{4}x^2} x^{\nu+n-\frac{1}{2}} \sum_{r=0}^{m} {}_mC_r \left(-x\right)^r D_{-n-2\nu-r}(x)$$

$$= \int_0^{\infty} e^{\frac{1}{4}t^2} t^{\nu+m-\frac{1}{2}} \left\{ \sum_{r=0}^{n} {}_nC_r \left(-t\right)^r D_{-m-2\nu-r}(t) \right\}$$

$$\times J_\nu(xt) \sqrt{xt} \, dt \quad (2.7)$$

and shows that

$$\psi(x) = \int_0^{\infty} \phi(t) \, J_\nu(xt) \sqrt{xt} \, dt.$$

To obtain the reciprocal relation, we simply interchange n and m and find that

$$\phi(x) = \int_0^\infty \psi(t)\, J_\nu(xt)\, \sqrt{xt}\, dt, \tag{2.8}$$

which proves the theorem enunciated above.

It may be noted that the restrictions imposed on n and m to be positive integers are not necessary but have been retained only for convenience and may easily be relaxed by safeguarding the convergence of the series and the integrals concerned.

3. In conclusion we consider a few special cases of this general result.

Thus when $m=n$; we find that the function

$$f(x) = e^{\frac{1}{4}x^2} x^{\nu+n-\frac{1}{2}} \sum_{r=0}^{n} {}_nC_r (-x)^r D_{-n-2\nu-r}(x)$$

is R_ν, i.e. self-reciprocal in J_ν-transform. (3.1)

For $n=0$ and $n=1$, this yields two well-known instances of self-reciprocal functions given by Varma,[3] viz.

$$f(x) = e^{\frac{1}{4}x^2} x^{\nu-\frac{1}{2}} D_{-2\nu}(x) \text{ is } R_\nu,\ R(\nu+\tfrac{1}{2}) > 0 \tag{3.2}$$

and

$$f(x) = e^{\frac{1}{4}x^2} x^{\nu+\frac{1}{2}} D_{-2\nu-3}(x) \text{ is } R_\nu,\ R(\nu+1) > 0. \tag{3.3}$$

Again taking $m=0$ and $n=1$ and applying the recurrence formula for the parabolic cylinder functions, we have[4]

$$\psi(x) = e^{\frac{1}{4}x^2} x^{\nu+\frac{1}{2}} D_{-2\nu-1}(x)$$

and

$$\phi(x) = (2\nu+1)\, e^{\frac{1}{4}x^2} x^{\nu-\frac{1}{2}} D_{-2\nu-2}(x),\ R(\nu+\tfrac{1}{2}) > 0, \tag{3.4}$$

which are Hankel-transforms of each other.

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A NEW PROOF OF THE CENTRE CIRCLE CHAIN

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De Longchamps(1) (1877), and Pesci(2) (1891) proved by elementary methods(3) the following chain of theorems:

THEOREM. *Given n lines in a plane, the centre-circles of sets of* $(n-1)$ *of these will be concurrent, also the centres of these will lie on a circle—the centre-circle of the n lines. The centre-circle of three lines is the circumcircle of the triangle formed by them.*

Morley(4) (1900) gave an independant proof using turns and map-equations of straight lines in the inversive plane. Grace(5) gave an attractive proof using the polar properties of isogonal curves. Later, White(6) (1925) and others pressed the geometry of higher dimensions for proving the same while Ramaswami Iyer and Bhimasenarao(6) (1926) proved the same by an elegant use of determinants and matrices.

In this paper we propose to establish the chain by a method similar to that used by Clifford for proving the chain of theorems bearing his name.

To this end we make use of a special type of curves C_n of class n touching the line at infinity $n-1$ times at points in equispaced directions, i.e. at points apolar to to the circular points I and J at infinity. Evidently then their tangential equation is of the form $u_n+u_{n+1}=0$, where u_r is a homogeneous expression of degree r in l and m. Also, since the points of contact given by $u_{n-1}=0$ are apolar to I, J, given by $l^2+m^2=0$, u_{n-1} is of the form $A(l+im^{n-1})+B(l-im^{n-1})$. Thus C_n has a freedom of order $n+1+2-1=n+2$ and so can be made to touch $n+2$ lines in general position. For special positions of the lines there will be not one but an infinity of C_n's touching them.

Since C_n is a curve of class n, having IJ as an $(n-1)$-ple tangent, there is just one tangent from I and one from J. Thus C has a unique focus. Consider now the locus of the foci of C_n's touching $n+1$ lines. Since a C_n can be made to touch $n+2$ lines in general, the tangents to the C_n's touching $n+1$ lines form two projective pencils. Thus we have the following

THEOREM 1. *The locus of the foci of C's touching $n+1$ lines is a circle.*

We shall call this circle the centre circle of the $n+1$ lines since there is always a C_n touching $n+2$ lines, the locus of the foci of C_n's touching any $n+1$ of these must pass through the focus of the C_n touching all the $n+1$ lines. So we have

THEOREM 2. *The centre circles of sets of $n+1$ of $n+2$ given lines are concurrent.*

Let P be any point on the centre circle of $n+1$ lines P_i, $(i=1, 2, .., n)$ and let S be its centre. Then there is a C_n touching P_i, PI and PJ. The tangents from J to this curve are IJ repeated $n-1$ times and PJ and similarly for I. Let now P approach and finally coincide with I. Then there must be a C_n touching the lines P_i, IJ and II, i.e. IS. For this C all the tangents from J coincide with IJ and so J must be one of the points of contact. But the points of contact are apolar to IJ and so all the points of contact coincide with J. Consider the C_{n+1} consisting of this C_n and J. (This is of class $n+1$, and touches the the line at infinity at the n coincident points J which, of course, are apolar to IJ.) Obviously SI, SJ and the lines l_i are tangents to this curve. Similarly, when P coincides with J, we will get a C_n which along with I constitutes another C_{n+1} touching the lines SI, SJ and the lines l_i. Thus there are two and therefore an infinity of C_{n+1}'s touching the lines l_i, SI and SJ. So if l_{n+2} is any other line, a C_{n+1} can be found to touch the $n+2$ lines l_i $(i=1, 2, .. n+2)$, and SI, SJ. In other words, if l_{n+2} is any

other line, and S is the centre of the centre circle of the lines l_i $(i = 1, 2, .., n+1)$, there is a C_{n+1} touching these $n+2$ lines and having S for a focus. We therefore have the following

THEOREM 3. *The centre circle of any $n+2$ lines will pass through the centre of the centre circle of any set of $n+1$ of these lines.*

Thus, with n lines, we have been able to associate, a centre circle in such a way that (1) the centre circles of any set of n of $n+1$ given lines are concurrent, and (2) the centres of these circles themselves lie on a circle the centre circle of the $n+1$ lines. Finally the centre circle of three lines is the locus of of foci of C_2's, i.e. parabolas touching them and so is the circumcircle of the triangle formed by them. Thus our centre circles are the same as the circles mentioned in (1).

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THE FIVE FAMILIES OF RULED SURFACES THROUGH A LINE OF A RECTILINEAR-CONGRUENCE.

BY

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1. Ogura[1], Hayashi[2] and Ram Behari[3] have proved by different methods that repeated applications of the operation of forming the Jacobian from Sannia's fundamental quadratic forms,[4]

$$f = E\,du^2 + 2\,F\,du\,dv + G\,dv^2$$
$$\phi = \delta\,du^2 + 2\,\delta'\,du\,dv + \delta''\,dv^2$$

lead to only five distinct families of ruled surfaces through the line l of the congruence, viz.

(1) Surfaces whose spherical representations are minimal lines,

(2) Developable surfaces,

(3) Surfaces of distribution,

(4) Principal surfaces and

(5) Characteristic surfaces, whose equations are respectively,

$$f = 0, \phi = 0, J(f, \phi) = 0, J[f, J(f, \phi)] = 0, J[\phi, J(f, \phi)] = 0.$$

The object of this paper is to verify the equations of developable surfaces, surfaces of distribution and principal surfaces already obtained by me in tensor form in a separate paper[5], by the method of dual coordinates and to obtain the equations of surfaces whose spherical representations are minimal lines and characteristic surfaces. An alternative method of finding the equation of characteristic surfaces and other forms of the equations of characteristic surfaces and principal surfaces with some of their important properties are also given.

2. Equations of surfaces whose spherical representations are minimal lines are given by

$$f \equiv g_{ij}\, du^i\, du^j = 0,$$

and the equations of developable surfaces are given by

$$\phi \equiv (\tfrac{1}{2}\bar{g}_{ij}\, du^i\, du^j) = 0,$$

where $f \equiv g_{ij}\, du^i\, du^j$ and $\phi = (\frac{1}{2}\bar{g}_{ij}\, du^i\, du^j)$ have been taken to be fundamental quadratic forms.(6)

Hence the equations of surfaces of distribution are

$$J(f, \phi) \equiv J(g_{ij}\, du^i\, du^j, \tfrac{1}{2}\bar{g}_{ij}\, du^i\, du^j) = 0$$

or
$$[g_{1i}\bar{g}_{2j} - g_{2i}\bar{g}_{1j}]\, du^i\, du^j = 0$$

or
$$e^{ks}\, g_{ki}\bar{g}_{sj}\, du^i\, du^j = 0$$

or
$$q_{ij}\, du^i\, du^j = 0.$$

Similarly the equations of the principal surfaces are given by

$$J[f, J(f, \phi)] \equiv J(g_{ij}\, du^i\, du^j, q_{ij}\, du^i\, du^j) = 0$$

or
$$[g_{1i}\,(q_{2j}+q_{j2}) - g_{2i}\,(q_{1j}+q_{j1})]\, du^i\, du = 0$$

or
$$e^{ks}\, g_{ki}\,(q_{sj}+q_{js})\, du^i\, du^j = 0.$$

The equations of characteristic surfaces are given by

$$J[\phi, J(f, \phi)] \equiv J(\tfrac{1}{2}\bar{g}_{ij}\, du^i\, du^j, q_{ij}\, du^i\, du^j) = 0$$

or
$$e^{ks}\, \bar{g}_{ki}\,(q_{sj}+q_{js})\, du^i\, du^j = 0.$$

3. We shall now give an alternative method of finding the equation of characteristic surfaces.

$$\frac{\text{Distance of the central point from the surface of reference}}{\text{Parameter of distribution}}$$

$$\equiv p = -\frac{q_{ij}\, du^i\, du^j}{\frac{1}{2}\bar{g}_{ij}\, du^i\, du^j}.$$

Differentiating the equation with respect to $\frac{du^1}{du^2}$ we get for extremum of p

$$[p\bar{g}_{ki} + q_{ki} + q_{ik}]\, du^i = 0, \quad [k = 1, 2].$$

Eliminating p between the above two equations we get

$$\bar{g}_{ki}\,(q_{sj}+q_{js})\, e^{ks}\, du^i\, du^j = 0,$$

which is the same as that obtained in §2.

4. We give other forms of the equations of principal and characteristic surfaces. We know that the parameter of distribution is given by

$$d = \frac{\phi}{f} = \frac{1}{2} \cdot \frac{\bar{g}_{ij}\, du^i\, du^j}{g_{ij}\, du^i\, du^j}.$$

The mean and total parameters are respectively

$$h = d_1 + d_2 = e^{ik}\, q_{ik}, \text{ and } k = \frac{1}{4} \frac{\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2}{e_{12}^2}.$$

For the principal surfaces, we have $J[f, J(f\, q)] = 0$, or from the theory of quadratic forms

$$\nabla f \cdot \phi - \tfrac{1}{2}\theta(f, \phi) \cdot f = 0,$$

where $\nabla f = e_{12}^2$ and $\theta(f, \phi) = e_{12}^2\, e^{ik}\, q_{ik}$.

Hence the equation is $\phi/f = \frac{1}{2}\, e^{ik}\, q_{ik}$, or $2d = h$.

Similarly for the characteristic surfaces, the form comes out to be

$$2k = hd.$$

5. We have found that the equations of characteristic surfaces are

$$\bar{g}_{ki}\, (q_{sj} + q_{js})\, e^{ks}\, du^i\, du^j = 0.$$

The condition for a normal congruence is

$$e^{ik}\, q_{ik} = 0 \text{ or } \bar{g}_{11}\, g_{22} + \bar{g}_{22}\, g_{11} - 2\bar{g}_{12}\, g_{12} = 0.$$

Applying this, we get

$$2[\bar{g}_{12}^2 - \bar{g}_{11}\, \bar{g}_{22}]\, g_{ij}\, du^i\, du^j = 0$$

or

$$g_{ij}\, du^i\, du^j = 0.$$

Hence if the congruence is normal, the characteristic surfaces coincide with the ruled surfaces whose spherical representations are minimal lines.

6. Some properties of ruled surfaces can be obtained from the forms of equations given in §4.

$2d = h$ is the equation of the principal surfaces.

If $h = 0$ this reduces to $d = 0$ which is the equation of developable surfaces.

But $h = 0$ is the condition for a normal congruence.

Hence for a normal congruence, the principal surfaces coincide with the developable surfaces.

The result in §5 can be obtained similarly by considering the equation of characteristic surfaees in the form $2k = hd$.

7. (a) From the equation of characteristic surfaces we find that they coincide with surfaces whose spherical representations are minimal lines if $2k = hd$ is the same as $1/d = 0$, $1/k = 0$, i.e. or $e_{12}^2 = 0$, i.e. the congruence is cylindrical.

(b) The characteristic surfaces coincide with the developable surfaces if the total parameter is zero.

But $$k = q_i^l q_k^s e^{ki} e_{sl} = -r^2.$$
Therefore the characteristic surfaces coincide with the developable surfaces if the foci of the congruence coincide.

(c) The characteristic surfaces coincide with the principal surfaces if $2k = hd$ is the same as $2d = h$, i.e. if $h^2 - 4k = 0$ or if $d_1 = d_2$, i.e. if the principal parameters are equal.

(d) The principal surfaces coincide with the surfaces whose spherical representations are minimal lines, if $1/h = 0$, or if the congruence is cylindrical.

8. We know that in a congruence of Bianchi the distance between the focal points is constant. Hence $r^2 \equiv q_i^l q_k^s e_{ls} e^{ki} =$ a constant.

But the total parameter $\equiv k = d_1 d_2 = q_i^l q_k^s e_{ls} e^{ik} =$ a constant.

Also in a congruence of Bianchi the distance between the limit points is constant, i.e.
$$(t_1 - t_2)^2 - 4r^2 \equiv (e^{ik} q_{ik})^2 = \text{a constant.}$$

But the mean parameter $\equiv h = e^{ik} q_{ik}$, hence $h =$ a constant.

Hence in a congruence of Bianchi both the total parameter and the mean parameter are constants.

Therefore both the characteristic surfaces and the principal surfaces reduce to the forms $d =$ constant.

Both these coincide with developable surfaces if the constants are zero.

I am grateful to Dr. Ram Behari for his guidance in this work.

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AN ASYMPTOTIC FORMULA IN PARTITIONS.

BY

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1. In what follows, $p(n, m)$ denotes the number of partitions of n into exactly m non-zero summands and $p_0(n)$ the number of unrestricted partitions of n.

Let $p_r(n) = \sum_{t=0}^{[n/r]} p_{r-1}(n-rt), r \geqslant 1, n \geqslant 1$;

and take

$$p_0(0) = p_r(0) = 1 \text{ and } p_r(-n) = 0, r \geqslant 1, n \geqslant 1.$$

Then, we have

$$p(n, m) = p_0(n-m) - p_1(n-2m-1) + \dots$$

$$+ (-1)^{r-1} p_{r-1}\left(n - rm - \frac{r(r-1)}{2}\right) + \dots \quad (1)$$

PROOF OF (1).

It is well known that $p(n, m)$ is the coefficient of x^n in the expansion of

$$x^m / \{ (1-x)(1-x^2) \dots (1-x^m) \}.$$

Moreover,

$$\{ (1-x)(1-x^2) \dots \}^{-1} = \sum_{t=0}^{\infty} p_0(t) x^t,$$

and

$$\{ (1-x)(1-x^2) \dots (1-x^r) \}^{-1} \sum_{t=0}^{\infty} p_0(t) x^t = \sum_{t=0}^{\infty} p_r(t) x^t.$$

Now,

$$(1-x^{m+1})(1-x^{m+2})(1-x^{m+3}) \dots\dots$$

$$= 1 - \frac{x^{m+1}}{(1-x)} + \frac{x^{2m+3}}{(1-x)(1-x^2)} - \dots\dots$$

$$+ (-1)^t \frac{x^{tm + \frac{t(t+1)}{2}}}{(1-x)(1-x^2) \dots\dots (1-x^t)} + \dots\dots$$

Hence

$$x^m/\{(1-x)(1-x^2)\dots(1-x^m)\}$$
$$= x^m\Big\{1 - \frac{x^{m+1}}{(1-x)} + \frac{x^{2m+3}}{(1-x)(1-x^2)} - \dots$$
$$+ (-1)^t \frac{x^{tm+\frac{t(t+1)}{2}}}{(1-x)(1-x^2)\dots(1-x^t)} + \dots\Big\}\Big\{\sum_{t=0}^{\infty} p_0(t)\, x^t\Big\}.$$

Equating the coefficients of x^n on the two sides, we get the result.

A formula similar to (1) was given by Erdos and Lehner* but it was not so handy.

2. For large values of n, Ramanujan showed that

$$p_0(n) \sim \frac{e^{2C\sqrt{n}}}{4n\sqrt{3}}, \text{ where } C = \frac{\pi}{\sqrt{6}}.$$

Hence

$$p_1(n) \sim \int_0^n p_0(n)\, dn \sim \frac{\sqrt{n}}{C} p_0(n),$$

$$p_2(n) \sim \tfrac{1}{2}\int_0^n p_1(n)\, dn \sim \frac{1}{2!}\left(\frac{\sqrt{n}}{C}\right)^2 p_0(n),$$

$$p_3(n) \sim \tfrac{1}{3}\int_0^n p_2(n)\, dn \sim \frac{1}{3!}\left(\frac{\sqrt{n}}{C}\right)^3 p_0(n),$$

and in general

$$p_r(n) \sim \frac{1}{r}\int_0^n p_{r-1}(n)\, dn \sim \frac{1}{r!}\left(\frac{\sqrt{n}}{C}\right)^r p_0(n). \tag{2}$$

3. Lemma A. *If n be large and $k = o(n^{\frac{3}{4}})$, then for any given r*

$$p_r(n-k) \sim p_r(n) \exp\left(-\frac{kC}{\sqrt{n}}\right). \tag{3}$$

We have

$$p_r(n-k) \sim \frac{1}{r!}\left(\frac{\sqrt{n-k}}{C}\right)^r \cdot \frac{\exp\left(2C\sqrt{n-k}\right)}{4(n-k)\sqrt{3}},$$
$$\sim \frac{1}{r!}\left(\frac{\sqrt{n}}{C}\right)^r \cdot \frac{1}{4n\sqrt{3}} \exp\left(2C\sqrt{n} - \frac{kC}{\sqrt{n}}\right);$$
$$\sim p_r(n) \exp\left(-\frac{kC}{\sqrt{n}}\right).$$

* *Duke Math. J.*, 8 (1941), 335-45.

LEMMA B. *If n be large and $k = o(n^{\frac{1}{2}})$, then for values of $m > \sqrt{n}$*,

$$p(n-k, m) \sim p(n, m) \exp\left(-\frac{kC}{\sqrt{n}}\right).$$

This follows from (1) and Lemma A.

4. We proceed to find an asymptotic formula for $p(n, m)$ when $m = \frac{\sqrt{n}}{C} \log \frac{\sqrt{n}}{x}$, where $x = O(1)$.

We have

$$p_0(n-m) \sim p_0(n) \exp\left(-\frac{mC}{\sqrt{n}}\right) \sim p_0(n) \cdot \frac{x}{\sqrt{n}};$$

$$p_1(n-2m-1) \sim p_0(n) \cdot \frac{\sqrt{n}}{C} \cdot \frac{x^2}{n};$$

$$p_2(n-3m-3) \sim \frac{1}{2!} p_0(n) \left(\frac{\sqrt{n}}{C}\right)^2 \cdot \left(\frac{x}{\sqrt{n}}\right)^3;$$

and so on. Hence from (1), we have

$$p(n, m) \sim p_0(n) \cdot \frac{x}{\sqrt{n}} \left\{1 - \frac{x}{C} + \frac{1}{2!}\left(\frac{x}{C}\right)^2 - \frac{1}{3!}\left(\frac{x}{C}\right)^3 + \cdots\right\}$$

$$\sim \frac{p_0(n)}{\sqrt{n}} x e^{-x/C}.$$

This was proved by Auluck, Chowla and the writer by the method of Erdos and Lehner in an earlier paper.*

The right side of (4) is maximal when $x = C$.

Hence $p(n, m)$ is maximal when

$$m \sim \frac{\sqrt{n}}{C} \log \frac{\sqrt{n}}{C}.$$

This holds even for small values of n. Thus for $15 \leqslant n \leqslant 300$, $p(n, m)$ is maximun when

$$m = \left[\frac{\sqrt{n}}{C} \log \frac{\sqrt{n}}{C} + \gamma\right], \text{ where } 2 \leqslant \gamma \leqslant 2.26.$$

Our conjecture that $p(n, m)$ has a unique maximum till remains unproved.

* *J. Indan Math. Soc.*, (2), 6 (1942), 105-12.

5. I* have shown that for $m > 3$, $n \geqslant m$,

$$\binom{n-1}{m-1} \leqslant m!\,p(n, m) \leqslant \binom{n-1+\frac{1}{4}m(m-1)}{m-1}.$$

Let $$p(n, m) = \frac{1}{m!}\binom{n-1+c_m(n)}{m-1}.$$

Consider the case when $n = m$ and m is large.
Then since $p(m, m) = 1$, we must have

$$(m-1)\log\left(m-1+c_m(m)\right) \sim \log\left\{m!\,(m-1)!\right\}.$$

This gives

$$c_m(m) \sim \frac{m(m-1)}{e^2}.$$

This suggests that a suitable form for $c_m(n)$ would be

$$\frac{m\,(m-1)}{4\,e^2}\left(1+\frac{m}{n}\right)^{2n/m}$$

This result could be used for determining the value of m for which $p(n, m)$ is maximal.

* *J. Bombay Univ.*, 11 (1942), 16-18.

ON THE DERIVATIVES OF INTEGRAL FUNCTIONS.*

BY

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1. The object of this paper is to investigate some inequalities concerning integral functions. Let $f(z)$ be an integral function of order ρ and lower order λ and let $f^{(s)}(z)$ be the sth derivative of $f(z)$ and let $M(r)$ $M^{(1)}(r),\ldots$ denote the maximum moduli of $f(z)$, $f^{(1)}(z),\ldots$ respectively, on the circle $|z|=r$. Let $n(r,f)=n(r)$ denote, as usual, the number of zeros of $f(z)$ lying in the circle $|z|=r$ and on its circumference and let $m(r)=\min\limits_{|z|=r}|f(z)|$.

The main results are given in the form of the following theorems.

THEOREM 1.†

The sequence

$$M(r),\ M^{(1)}(r),\ldots\ldots\ldots,\ M^{(s)}(r),$$

where $r>r_0=r_0(f)$ forms an increasing sequence, provided that $f(z)$ is an integral function of lower order λ, $\lambda>1$.

THEOREM 2.

The sequence

$$M(r),\ M^{(1)}(r),\ldots;\ M^{(s)}(r),$$

where $r>r_0=r_0(f)$ forms a decreasing sequence provided that $f(z)$ is of order less than $\frac{1}{2}$.

* I wish to thank Dr. R. S. Varma for suggesting this problem and for his guidance and also Dr. S. M. Shah for his helpful suggestions.

† I am indebted to the referee for the improvement suggested in Th. 1.

r_0 is not necessarily the same at each occurence.

THEOREM 3.

$$M^{(s)}(r) > M(r) \left\{\frac{\log M(r)}{r \log r}\right\}^s \quad (s = 1, 2 \ldots),$$

for $r > r_0$ under the condition of Theorem 1.

2. PROOF OF THEOREM I.

We consider the inequality of Valiron*

$$M^{(1)}(r) \sim M(r)\ N(r)/r \text{ for } r > r_0. \qquad (2.1)$$

If in (2.1), we replace $M(r)$, $M^{(1)}(r)$, $N(r)$ by $M^{(1)}(r)$ $M^{(2)}(r)$, $N^{(1)}(r)$ respectively, which is possible, we get

$$M^{(2)}(r) \sim M^{(1)}(r)\ N^{(1)}(r)/r \text{ for } r > r_0. \qquad (2.2)$$

Similarly

$$M^{(s)}(r) \sim M^{(s-1)}(r)\ N^{(s-1)}(r)/r \text{ for } r > r_0. \qquad (2.3)$$

Since the lower order of $f^{(1)}(z)$ is the same as that of $f(z)$ as may easily be seen from the following inequalities†

$$\frac{M(r) - |f(0)|}{r} \leqslant M^{(1)}(r) \leqslant \frac{M(R)}{R-r},$$

we can establish similar inequalities for $M^{(2)}(r)$ and $M^{(1)}(r)$ and others and prove that the lower order of the derivatives is the same as that of the given function. Further Whittaker‡ has shown that if

$$\varliminf_{r\to\infty} \frac{\log \log M(r)}{\log r} = \lambda,$$

then

$$\varliminf_{r\to\infty} \frac{\log N(r)}{\log r} = \lambda.$$

If $f(z)$ is of lower order λ, $\lambda > 1$, then from (2.1), (2.2) and (2.3) we get

$$M(r) < M^{(1)}(r) < \ldots < M^{(s)}(r) \text{ for } r > r_0.$$

* Valiron: *Theory of Integral Functions*, p. 105.

† Titchmarsh: *The Theory of Functions*, p. 265.

‡ Whittaker: *Journ. Lond. Math. Soc.*, 8, (1933), 20.

3. PROOF OF THEOREM 2.

We consider the inequality*

$$M^{(1)}(r) < M(r)\, r^{2\rho-1+\epsilon} \text{ for } r > r_\epsilon.\dagger \qquad (3.1)$$

If we proceed exactly as Valiron has done with $M^{(2)}(r)$ we have for $r > r_\epsilon$,

$$M^{(2)}(r) < U^{(2)}(r).\, r^{\rho+\epsilon},$$

where $U^{(2)}(r)$ is the modulus of the maximum term in the Taylor expansion of $f^{(2)}(z)$, for the order of the derivative is the same ; but

$$U^{(2(}(r) = N(2)\, C_{N(2)}.\, r^{N(2)-1}\ddagger$$
$$\leqslant \frac{N(2)\ U^{(1)}(r)}{r}$$

and so

$$M^{(2)}(r) < U^{(1)}(r).\, r^{2\rho-1+\epsilon} < M^{(1)}\, r^{2\rho-1+\epsilon} \qquad (3.1)$$

Similarly

$$M^{(s)}(r) < M^{(s-1)}(r).\, r^{2\rho-1+\epsilon} \qquad (3.2)$$

From (3.1) and (3.2) for all large r, we have

$$M(r) > M^{(1)}(r) > \ldots > M^{(s)}(r), \text{ provided } \rho < \tfrac{1}{2}.$$

4. PROOF OF THEOREM 3.

We know that§

$$M^{(1)}(r) \geqslant \frac{M(r) \log M(r)}{r \log r}$$

and if $\lambda > 1$, we know that $M^{(1)}(r) > M(r)$.

Also

$$M^{(2)}(r) > \frac{M^{(1)}(r) \log M^{(1)}(r)}{r \log r}$$
$$> \frac{M(r)}{r} \cdot \frac{\log M(r)}{\log r} \cdot \frac{\log M^{(1)}(r)}{r \log r}$$
$$> M(r) \left\{\frac{\log M(r)}{r \log r}\right\}^2.$$

* Valiron : *loc. cit*, p. 35.

† ϵ is chosen so small that $2\rho - 1 + \epsilon < 0$.

‡ $N(2)$ and $C_{N(2)}$ denote the rank and the coefficient of the maximum term in the Taylor expansion of $f^{(2)}(z)$.

§ T Vijayaraghavan, *Journ. Lond. Math Soc.*, 10 (1935), 116.

Similarly

$$M^{(s)}(r) \geqslant M(r)\left\{\frac{\log M(r)}{r\log r}\right\}^s$$

for $r > r_0(f)$ $s = 1, 2, \ldots$

5. We offer a few remarks on Theorems 1 and 2 by considering some examples.

(i) The function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha} \quad \alpha \geq 0$$

is of order $1/\alpha$.

It is easy to see that if $\alpha < 1$, $M(r)$, $M^{(1)}(r), \ldots$ form an increasing sequence, while if $\alpha > 1$ they form a decreasing sequence and for $\alpha = 1$, $M(r) = M^{(1)}(r) = \ldots$

Theorem 2, however, does not say anything about functions, whose order ρ is given by $\frac{1}{2} < \rho < 1$.

(ii) $f(z) = e^{kz}$ which is of order 1.

It is easy to see that when $k < 1$, $M(r)$, $M^{(1)}(r), \ldots$ form a decreasing sequence; if $k > 1$, they form an increasing sequence and if $k = 1$, $M(r) = M^{(1)}(r) = \ldots$

The above shows that the maximum moduli of the various derivatives of an integral function of order one may either form an increasing or decreasing sequence, or they may be equal.

THE ORDINAL CORRESPONDENCE AND CERTAIN CLASSES OF MULTIPLICATIVE FUNCTIONS OF TWO ARGUMENTS.*

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1. INTRODUCTION.

The object of this paper is first to establish an important reversible one-to-one correspondence between *general* multiplicative functions(1) and the ordinal functions.† This correspondence may be referred to as *the ordinal correspondence*.‡ A remarkable feature of the ordinal correspondence is that it leaves *invariant*, the product and the composite of two multiplicative functions, as well as, the inverse of any function. The ordinal correspondence leads us to a finer classification of the ordinal functions by means of an associated index. This classification of ordinal functions gives rise to a parallel classification of the modular functions. The ordinal correspondence also leads us to a generalization of the principal functions by introducing an index. The principal functions so generalized, may be called *Quasi-principal* functions.

1.1. I have shown elsewhere(2) that every multiplicative function $F(M, N)$ of two arguments satisfies an identical equation of the form,

$$F(M, N) = F(M, 1)\, F(1, N) \,.\, C(M, N),$$

where $C(M, N)$ is a cardinal function determined by and called the *cardinal component* or briefly the c-component of

* Most of the results of this paper were given in a thesis, approved for the M.Sc Degree of the University of Madras and I am much indebted to Dr. R. Vaidyanathaswamy for his guidance.

† The definitions and notations, unless otherwise explained, are the same as in (1). The references at the end of the paper will be quoted by their numbers.

‡ The correspondence is suggested in (1) pp. 657-58, but has not been systematically studied.

$F(M, N)$. If $F(M, N)$ is an ordinal or modular function, the c-component is also ordinal.* In this paper we shall determine the index of this c-component.

1.11. It is also shown[2] that Ramanujan's† function[3] $C_M(N)$ is a modular function with the modulus M and that its identical equation is given by

$$C_M(N) = C_M(1)\ C_1\ (N).\ \text{princ I} = \mu(M)\ E(N).\ \text{princ I}.$$

This relation shows that in general we can recognise an important special class of modular functions, viz. those whose *c-components* are *principal* functions. It is appropriate to refer to such functions as *functions of Ramanujan's class*. The concept of quasi-principal functions suggests at once, a generalization of functions of Ramanujan's class, by substituting quasi-principal for principal functions. At the end of this paper, we derive some important properties of the generalized functions of Ramanujan's class and show how they can be generated.

2. The ordinal correspondence.

2.1. Let $F(M, N)$ be any multiplicative function of M, N. We can define a function $f(M, N)$ thus:—

$$f(M, N) = F(M/N, N) \text{ if } N \text{ divides } M,$$
$$\text{and} \qquad = 0 \text{ otherwise.}$$

We may denote f symbolically as $O\ (F)$, where O is called the ordinal operator which transforms F into f.

The function $f(M, N)$ so defined is an *ordinal function with the major argument M.*

Proof. We have only to show that f is multiplicative in M, N; (for, the ordinal character is implied in the definition itself).

Let M, M', N, N' be integers such that

$$(MN, M'N') = 1.$$

Then $(M, M') = (M, N') = (N, M') = (N', N') = 1.$

* Vide (2), p. 524.

† It may be mentioned here that Ramanujan's function has been shown to be identical[4] with Von Sterneck's function[5].

Now either* $NN'|MM'$ or $NN'\times$† MM'. If $NN'|MM'$, then from (2.1) it follows that $N|M$ and $N'|M'$.

Therefore from the definition of f

$$f(M, N)f(M', N') = F(M/N, N)\,F(M'/N', N')$$
$$= F(MM'/NN', NN')$$

since F is multiplicative

$$= f(MM', NN').$$

On the other hand if $NN'\times MM'$, it is necessary by (2.1) that either $N\times M$ or $N'\times M'$. Therefore either $f(M, N) = 0$ or $f(M', N') = 0$.

Hence $f(M, N)f(M' N') = 0 = f(MM', NN')$.

Thus $f(M, N)$ is multiplicative in M, N.

The correspondence is reversible, for we can go back from $f(M, N)$ to $F(M, N)$ thus:—

$$f(M, N) = F(M/N, N) \text{ if } N|M$$
$$= F(M', N), \text{ writing } M/N = M'$$

or $$F(M', N) = f(M'N, N).$$

Hence $F(M', N)$ is known for all values of M', N, so that it is completely determined by f.

We may write $F = O^{-1}(f)$ to denote the inverse operation.

3. Ordinal functions associated with an index.

3·1. It is clear that we can repeat the operation O successively any number of times. Thus we have $f_2 = O(f_1) = O[O(F)]$, where f_1 is written for $O(F)$. We may write $f_2 = O^2(F)$; more generally $f_r = O^r(F)$ can be defined inductively by $O^r(F) = O[O^{r-1}(F)]$.

It is easy to show that,

$$O^{r+s}(F) = O^r[O^s(F)] = O^s[O^r(F)].$$

Similarly we may write $F = O^{-r}(f_r)$ to denote the inverse operation repeated r times.

It can be seen that $O^r(F)$ is an ordinal function which vanishes if $N^r\times M$. Any number r with this property is called *an ordinal index*, (or briefly *an index*), of the function.

* (a, b) denotes the greatest common divisor of a, b.

† The symbol $\times$ is here used for 'does not divide'.

Now if $0 < s < r$, $O^r(F)$ vanishes whenever $N^s \times M$, for this implies $N^r \times M$. Thus if r is an index, any number smaller than r is also an index. Therefore, we define *the highest number r with the above property as the precise ordinal index* (or sometimes referred to briefly as *the index*) *of the ordinal function.* Thus if s is an index of an ordinal function, its precise ordinal index $\geqslant s$.

Let $f(M, N)$ be an ordinal function with any index r.

Then the property $f(M, N) = 0$ if $N^r \times M$ implies and is implied by the property $f(M, N) = 0$ if $N^r > M$.

For, whenever $N^r > M$, $N^r \times M$. Hence the first part follows. To show that $f(M, N) = 0$ if $N^r > M$, implies $f(M, N) = 0$ if $N^r \times M$, it would be sufficient to prove that $f(p^m, p^n) = 0$ if $p^{nr} \times p^m$, since f is multiplicative, (p being any prime). This is obvious, since $f(p^m, p^n) = 0$ if $p^{nr} > p^m$ and $p^{nr} > p^m$ implies and is implied by $p^{nr} \times p^m$.

3.11. It follows from the above that a necessary and sufficient condition for a number r to be the precise ordinal index is (i) $f(M, N) = 0$ whenever $N^r > M$ and (ii) there is at least a pair of numbers M, N, satisfying $N^{r+1} > M$ such that $f(M, N) \neq 0$.

3.2. *The ordinal functions with the index infinity.* There are ordinal functions with the index $\geqslant$ any *arbitrarily assigned number.* Such functions may be said to have *the index infinity.* We may easily specify all such functions :—

The only ordinal functions $f(M, N)$ with the major argument M and the index ∞ are of the form $\psi(M) E_0(N)$, where $\psi(M) = f(M, 1)$.

For if r is any integer, $\psi(M) E_0(N)$ has the index $\geqslant r$ since when $N^r > M, N > 1$ and the function vanishes. Conversely if $f(M, N)$ has the index ∞, $f(M, N) = 0$ when $N > 1$ since we can then find an r such that $N^r > M$; hence $f(M, N) = f(M, 1) E_0(N)$.

As an important particular case of the above result, we may mention that since $E_0(M, N) = E_0(M) E_0(N)$ it has the index ∞.

(It is clear from the definition of the precise ordinal index of an ordinal function that a non-ordinal function may be regarded as a function with the index o. Thus any multiplicative function in general can be said to have the index $\geqslant 0$.)

We next prove the following

3·3. THEOREM. *The ordinal correspondence increases the index of any function by unity.*

PROOF. Suppose $f = O(F)$, where F has the index r. Then $F(M, N) = 0$ if $N^r > M$ and there is a pair of numbers M, N for which $N^{r+1} > M$ such that $F(M, N) \neq 0$.

Now $f = O(F) = F(M/N, N)$ if $N | M$ and o otherwise.

But $F(M/N, N) = 0$ if $N^r > M/N$ (i.e.) $N^{r+1} > M$.

Hence $f(M, N) = 0$ has an index $(r+1)$.

To show that $(r+1)$ is the precise ordinal index of f, suppose $s > (r+1)$ is the index of f. Then $f(M, N) = 0$ if $N^s > M$. Therefore $F = O^{-1}(f) = f(MN, N) = 0$ if $N^s > MN$, or $N^{s-1} > M$. But $(s-1) \geqslant (r+1) > r$. Hence the index of F is not r, contrary to the supposition.

Therefore $f(M, N)$ has the index $(r+1)$.

COR. (i) It follows that $O^t(F)$ *has the index* $(r+t)$.

From (i) we deduce

COR. (ii) *An ordinal function with the index* ∞ *corresponds to itself; in particular* $E_0(M, N)$ *corresponds to itself and* $O^t E_0(M, N) = E_0(M, N)$.

4. QUASI-PRINCIPAL FUNCTIONS.

A principal function $P(M, N)$ is clearly a special type of ordinal function with the major argument M; for, $P(M, N) = 0$ if $M \neq N$, necessarily implies $P(M, N) = 0$ if $N > M$. Regarding its index, we note that if $P(M, N) \neq E_0(M, N)$ (which is evidently a principal function), its index is unity. For, if $P(M, N)$ has the index r (> 1), then $P(M, M) = 0$ if $M \neq 1$, since $M^r > M$. Hence $P(M, N) = E_0(M, N)$.

If in the ordinal correspondence we take $F(M, N) = P(M, N)$, where $P(M, N) = \text{princ } \psi$, then $O(F) = P(M/N, N)$ if $N \mid M$ and o otherwise.

But $P(M/N, N) = \psi(N)$ if $M \mid N = N$, and o otherwise.

Hence $\qquad O(F) = 0$ if $M \neq N^2$

and $\qquad\qquad = \psi(N)$ otherwise.

Proceeding similarly we arrive at the function $O^{r-1}(F)$ which vanishes if $M \neq N^r$ and is equal to $\psi(N)$ otherwise.

We call $O^{r-1}(F)$, *a quasi-principal function with the index r and the major argument M, equiavalent to* $\psi(N)$ and we write

$$O^{r-1}(F) = \text{quasi-princ } \psi.$$

Since $P(M, N)$ has the index unity in general, it follows from (3.3) that $O^{r-1}(F)$ is an ordinal function with the index r.

A quasi-principal function $Q_r(M, N)$ of index r with the major argument M, is obviously a *cardinal function* since $D_M(Q_r) = E_0(N)$, and $D_N(Q_r) = E_0(M)$. It follows at once by (1.1) *that the c-component of a quasi-principal function is identical with the function itself.*

5. THE ORDINAL FUNCTIONS WHICH CORRESPOND TO THE ELEMENTARY FUNCTIONS IN THE ORDINAL CORRESPONDENCE.

We have already seen that the elementary function $E_0(M, N)$ corresponds to itself. As further illustrations of the ordinal correspondence, we shall find the ordinal functions which correspond to the elementary functions $E(M, N)$, $E_0(M)$ $E(N)$ and $\rho(M, N)$.

5.1. Let $F(M, N) = E(M, N) = 1$ for all values of M, N. Then

$$O(E) = E(M/N, N) \text{ if } N \mid M \text{ and o otherwise;}$$
$$= 1 \text{ if } N \mid M \text{ and o otherwise.}$$

Hence* $O(E) = \delta(M, N)$. Let us write $\delta_1(M, N)$ for $\delta(M, N)$. Then

* See (2), 525.

$_2(M, N) = O(\delta_1) = \delta_1(M/N, N)$ if $N \mid M$ and o otherwise,

$= 1$ if $N \mid M/N$ and o otherwise,

$= 1$ if $N^2 \mid M$ and o otherwise.

Proceeding similarly we deduce that

$\delta_r(M, N) = O^r(E)$ vanishes if $N^r \mid M$ and is 1 otherwise.

The index of $\delta_r(M, N)$ is r by 3.3 Cor. (i), since $E(M, N)$ is clearly a non-ordinal function (i.e. a function with the index o).

Thus $\delta_r(M, N)$ *is a typical elementary Ordinal function with the index r.*

5.2. Suppose $F(M, N) = E_0(M)\, E(N)$.

Then $O(F) = E_0(M/N)\, E(N)$ if $N \mid M$ and o otherwise;

$= 1$ if $M/N = 1$ (i.e.) if $M = N$ and o otherwise. Or

$$O(F) = \text{princ } E.$$

It follows that $O^r(F) = \text{quasi-princ}_r\, E(N)$.

5.3. Next let us take $F(M, N) = \rho(M, N)$, where $\rho(M, N)$ is Kronecker's function,(6) defined thus:— $\rho(M, N) = 1$ if $(M, N) = 1$ and o otherwise.

It is clear that $\rho(M, N) = E_0(g)$, where $g = (M, N)$ and hence is multiplicative in M, N.

Let $O(\rho) = \rho_1(M, N)$. Then

$\rho_1 = \rho(M/N, N)$ if $N \mid M$ and o otherwise;

$= 1$ if $(N, M/N) = 1$ and o otherwise;

$= 1$ if N is a block factor of M and o otherwise.

Therefore $\rho_2 = O^2(\rho) = \rho_1(M/N, N)$ if $N \mid M$ and o otherwise,

$= 1$ if N is a block factor of M/N and o otherwise,

$= 1$ if N^2 is a block factor of M and o otherwise.

Proceeding in this way we arrive at $O^r(\rho) = \rho_r(M, N)$ which vanishes if N^r is not a block factor of M and is 1 otherwise.

Thus $\rho_r(M, N)$ is a special type of ordinal function and has the index r, since ρ has the index 0.

6. We now establish that the ordinal correspondence leaves *invariant* the product, the composite and the inverse.

6.1. THEOREM. *If* $f_i = O(F_i), i = 1, 2,$ *and*

$$f(M, N) = f_1(M, N) \times f_2(M, N),$$
$$F(M, N) = F_1(M, N) \times F_2(M\ N),$$

then

$$f = O(F).$$

For if $N \mid M, f(M, N) = f_1((M, N) \times f_2(M, N)$

$$= F_1(M/N, N)\ F_2(M/N, N)$$
$$= F(M/N, N)$$

and $= 0$ otherwise.

Hence $f = O(F)$ and thus the ordinal correspondence preserves multiplication.

6.11. COR.

$$O^r[F_1(M, N) \times F_2(M, N] = O^r[F_1(M, N)] \times O^r[F_2(M, N)].$$

6.2. THEOREM. *If* $f_i = O(F_i),\ i = 1, 2$ *and*

$$f = (f_1 . f_2)\ (M, N),\ F = (F_1 . F_2)\ (M, N),$$

then

$$f = O(F).$$

PROOF. We first show that $f(M, N)$ is an ordinal function with the major argument M.

$$f(M, N) = f_1(M, N) . f_2(M, N)$$
$$= \sum_{d_i/M} \sum_{\delta_j/N} f_1(M/d_i, N/\delta_j)\ f_2(d_i, \delta_j).$$

Suppose $N \chi M$. Then either $N/\delta_j \chi M/d_i$ or $\delta_j \chi d_i$.

Therefore every term on the right side of $f(M, N)$ vanishes.

Hence f is an ordinal function with the major argument M.

Let $N | M$. Then if $\delta_j | d_i$, $N/\delta_j | M/d_i$.

Hence $f(M, N) = \Sigma \Sigma f_1 (M/d, N/\delta) f_2 (d, \delta)$, summed for such divisors d of M, δ of N, satisfying $\delta | d$. Or

$$f(M, N) = \sum \sum F_1 \left(\frac{M/d}{N/\delta}, \frac{N}{\delta} \right) F_2 \left(\frac{d}{\delta}, \delta \right)$$

$$= \sum \sum F_2 \left(\frac{M/N}{d/\delta}, \frac{N}{\delta} \right) F_2 \left(\frac{d}{\delta}, \delta \right)$$

$$= \sum \sum F_1 \left(\frac{M/N}{\gamma}, \frac{N}{\delta} \right) F_2 (\gamma, \delta), \qquad \text{(I)}$$

where $\gamma = d/\delta$ is a divisor of M/N and δ/N.

Now

$$F(M, N) = F_1 (M, N) . F_2 (M, N)$$

$$= \sum_{d_i/M,} \sum_{\delta_j/N} F_1 (M/d_i, N/\delta_j) F_2 (d_i, \delta_j).$$

Hence

$$F\left(\frac{M}{N}, N\right) = \sum \sum F_1 \left(\frac{M/N}{\alpha}, \frac{N}{\beta} \right)$$

$$\times F_2 (\alpha, \beta), \text{ where } \alpha \left| \frac{M}{N}, \beta \right| N. \qquad \text{(II)}$$

It can be easily seen that any term of I gives rise to a single term of II and conversely. Hence I and II are identical.

Therefore $f(M, N) = F(M/N, N)$ if $N | M$ and 0 otherwise.

Hence $f = O(F)$ and thus the ordinal correspondence leaves invariant the composite.

6.21. Cor. $O^r(F_1.F_2) = O^r(F_1), O^r(F_2)$.

6.22. Remark. Incidentally, we have shown that "the composite of two ordinal functions is an ordinal function".

6.3. Theorem. *If $f = O(F)$, then $f^{-1} = O(F^{-1})$.*

For, if $g = O(F^{-1})$,

$$f.g = O(F.F^{-1}) \text{ by Th. 6.2}$$

$$= O(E_0) = E_0(M, N) \text{ by 3.3. Cor. (ii).}$$

Hence $g = f^{-1} = O(F^{-1})$. Thus the inverse of any function is left invariant by the ordinal correspondence.

6.31. We can easily deduce

COR. $O^r(F^{-1}) = [O^r(F)]^{-1}$.

6.32. NOTE. Incidentally we have established the result "the inverse of an ordinal function is an ordinal function".

6.33. We also have

The composite of a non-ordinal and an ordinal function is non-ordinal.

For, if $f = f_1 . f_2$ is ordinal, (where f_1 is a non-ordinal and f_2 an ordinal function), $f_1 = f . f_2^{-1}$ must be ordinal by 6.32 and 6.22 which contradicts the hypothesis.

7. PROPERTIES OF ORDINAL FUNCTIONS AND CLASSIFICATION OF MODULAR FUNCTIONS.

7.1. *The derivate of an ordinal function $f(M, N)$ with the index r with respect to the major argument M is $E_0(N)$.*

For, $D_M(f) = f(1, N)$. But $f(1, N) = 0$ if $N^r > 1$ $(r > 0,\ N \neq 1)$.

Hence

$$D_M(f) = E_0(N).$$

It follows that :—

7.11. A necessary condition for a function $f(M, N)$ to be an ordinal function with the major argument M is that $D_M(f) = E_0(N)$.

7.12. The only ordinal functions $f(M, N)$ with the major argument M, which are products of two functions of M, N respectively are of the form $\psi(M)\, E_0(N)$, where $\psi(M) = f(M, 1)$. Thus they are precisely the functions with the index ∞. (Compare 3.2).

7.2. *The integral with respect to the minor argument N of the ordinal function $f(M, N)$ with the precise ordinal index r.*

We now establish an important property of ordinal functions, which leads to the parallel classification of the modular functions, referred to in the introduction. It will

be shown that if $F(M, N)$ is the integral of $f(M, N)$ w.r.t. N, then F possesses the remarkable property of having M_r (the greatest integer whose rth power divides M) as an *effective* modulus (by which is meant, that the value of $F(M, N)$ is unaltered if N is increased by any multiple of M_r). The number r will be called a modular index of the modular function $F(M, N)$ with the modulus M. It is easy to see that if r is a modular index, s $(0 < s < r)$ is also a modular index. *The highest number with the above property* will be called the *precise modular index* (or briefly the p.m. index) of $F(M, N)$. It follows that the p.m. index $\geqslant s$ if s is a modular index.

7.21. Thus a necessary and sufficient condition for an integer r to be the p.m. index is (i) $F(M, N) = F(M, N+\lambda M_r)$ where λ is any integer and (ii) $F(M, N) \neq F(M, N+\lambda M_{r+1})$, for a pair of values of M, N and an integer λ, M_{r+1} being the greatest number whose $(r+1)$th power divides M. (It may be mentioned that we always refer to $F(M, N)$, *as the modular function F with the modulus M, since whatever may be the index, M is always a modulus*; and specify the index r if $r > 1$, to indicate the effective modulus.)

As a simple example of a modular function with the p.m. index r, we have $F(M, N) = g_r$, where $g_r = (M_r, N)$, M_r having the same meaning as before. M_r is an effective modulus since $(M_r, N) = g_r = (M_r, N+\lambda M_r)$. The p.m. index of F is r since $F(p^r, p^t) = p$, $(t \geqslant 1) \neq F(p^r, p^t+1)$ which is 1, (taking $\lambda = 1$ and noting that $M_r = p$, $M_{r+1} = 1$). More generally any multiplicative function $\psi(g_r)$ of $g_r = (M_r, N)$ has the p.m. index $\geqslant r$.

7.3. THEOREM. *The integral w.r.t. the minor argument N of any ordinal function $f(M, N)$ with the index r and the major argument M is a modular function with the modulus M and the precise modular index r.*

PROOF. Let $F(M, N)$ be the integral of f, w.r.t. N.

Then
$$F(M, N) = f(M, N) . E_0(M) E(N)$$
$$= \sum_{d/N} f(M, d)$$
$$= \sum_{d^r/M,\ d/N} f(M, d)$$
since $f(M, d) = 0$ if $d^r \nmid M$.

Let $M = \prod_{i=1}^{k} p_i^{m_i}$; $N = \prod_{i=1}^{k} p_i^{n_i}$ $(m_i \geqslant 0,\ n_i \geqslant 0)$. Then $M_r = \prod_{i=1}^{k} p_i^{[m_i/r]}$ (where $[m_i/r]$ = the greatest integer in m_i/r) is clearly the greatest number whose rth power divides M. Now $g_r = (M_r, N) = \prod_{i=1}^{k} p_i^{\min\,([m_i/r],\ n_i)}$. If $d \mid g_r$, then $d \mid M_r, N$ and therefore $d \mid M, N$. Further $d^r \mid M_r^r$ and so $d^r \mid M$. Conversely if $d \mid N$ and $d^r \mid M$, then $d \mid g_r$. Hence
$$F(M, N) = \sum_{d/g_r} f(M, d) = F(M, g_r).$$

Therefore M_r is an effective modulus of $F(M, N)$.

To show that F has the p.m. index r, we require the following theorem which is the converse of 7.3.

7.4. THEOREM. *The composite of a modular function $F(M, N)$ with the modulus M and the precise modular index r, and $E_0(M) E^{-1}(N)$ is an ordinal function with the index r and the major argument M.*

PROOF. Let
$$f(M, N) = F(M, N) . E_0(M) E^{-1}(N)$$
$$= \sum_{d/M} F(M, d)\, \mu(N/d).$$

Because f is multiplicative, it would be sufficient if we show that $f(p^m, p^n) = 0$ whenever $p^{nr} \nmid p^m$ for a prime p.

Now $f(p^m, p^n) = F(p^m, p^n) - F(p^m, p^{n-1})$.

Let $nr > m$; then $n > m/r \geqslant [m/r]$ and $n-1 \geqslant [m/r]$.

Therefore, since F has the p.m. index r
$$F(p^m, p^{[m/r]}) = F(p^m, p^{[m/r]} + \lambda p^{[m/r]})$$

$= F(p^m, p^n)$ taking $\lambda = p^{n-[m/r]} - 1$

$= F(p^m, p^{n-1})$ taking $\lambda = p^{(n-1)-[m/r]} - 1$.

Hence $f(p^m, p^n) = 0$ if $p^{nr} \chi p^m$.

Therefore $f(M, N) = 0$ if $N^r \chi M$.

The precise indices of f and F. Combining Theorems 7.3 and 7.4, it is clear that if f has the precise ordinal index r, F has the precise modular index r and conversely.

The primitive of a modular function:—From 7.3 and 7.4, we also see that corresponding to a given modular function F, there exists an ordinal function f, such that F is the integral of f w.r.t. the minor argument. We call the ordinal function $f(M, N)$, *the primitive* of the modular function $F(M, N)$.

We have incidentally proved*

7.5. THEOREM. *If $F(M, N)$ is a modular function with the modulus M and the precise modular index r, then*

$$F(M, N) = F(M, g_r),$$

g_r and M_r having the same meaning as before.

7.6. The following properties of a modular function $F(M, N)$ with the modulus M and the p.m. index r, readily follow from the corresponding properties of its primitive $f(M, N)$:—

7.61. (*a*) $D_M(F) = E(N)$; (*b*) $D_N(F) = D_N(f)$.

From (*a*) we deduce the following:

(*c*) A necessary and sufficient condition for a function $F(M, N)$ to have the modulus M is $D_M(F) = E(N)$;

(*d*) A modular function $F(M, N)$ with the modulus M, is multiplicative in M alone† ;

(*e*) The composite of two modular functions cannot be a modular function; and

(*f*) The inverse of a modular function cannot be a modular function.

* See 7.31 or 7.41.

† See (2), p. 521.

7.62. The only modular functions $F(M, N)$ which are products of two functions of M, N respectively are of the form $\psi(M)\,E(N)$, where $\psi(M) = F(M, 1)$; these are precisely the functions which have the modular index ∞.

7.7. *The modular function corresponding to* $\delta_r\ (M, N)$.

We derive the integral of the typical elementary ordinal function, $\delta_r(M, N) = 1$ if $N^r \mid M$ and 0 otherwise.

Now

$$\delta_r(M, N)\,.\ E_0(M)\,E(N)$$

$$= \sum_{d/N} \delta_r(M, d) = \sum_{d^r/M,\, d/N} \delta_r(M, d) = \sum_{d/g_r} \delta_r(M, d)$$

$= d(g_r)$ the total number of divisors of $g_r = (M_r, N)$.

Thus $d(g_r)$ *is the modular function corresponding to* $\delta_r(M, N)$.

8. The indices of the product, composite and the inverse of ordinal functions.

8.1. Theorem. *If* $f_r(M, N)$ *and* $\psi_s(M, N)$ *are ordinal functions with the major argument M and the indices r, s respectively, their product has the index* $\geqslant s$, *where* $s \geqslant r$.

Proof. Let $f(M, N) = f_r(M, N)\ \ \psi_s(M, N)$;

$$f_r = O^r(F_1),\ \psi_s = O^s(F_2),$$

where F_1, F_2 are non-ordinal since r, s are the precise indices of f, ψ. Then by 3.1

$$\begin{aligned} f(M, N) &= O^r(F_1) \times O^s(F_2) \\ &= O^r(F_1) \times O^r[O^{s-r}(F_2)] \\ &= O^r(F_1) \times O^r F_3, \text{ where } F_3 = O^{s-r}\ (F_2) \\ &= O^r(F_1 \times F_3) \text{ by } (6.11) \\ &= O^r(F) \text{ where } F = F_1 \times F_3. \end{aligned}$$

Let $s - r = t\ (\geqslant 0)$. Since F_1 is non-ordinal and F_3 has the index t, it easily follows that F has the index $\geqslant t$.

Therefore by 3.3 (i) f has the index $\geqslant (r+t) = s$.

Note. This is the best possible result, for we cannot say in general that F has the index t, since even though there is a pair of values of M, N satisfying $N^{t+1} > M$ such that $F_3(M, N) \neq 0$, F may still vanish because, for this

particular pair of values the non-ordinal function F_1 may vanish. The following example will illustrate the above theorem :—

Let $G(M, N) = E_0(M, N)$ for the prime base p
and $= \delta_r(M, N)$ for all other bases;
and $H(M, N) = \delta_s(M, N)$ for the prime base p
and $= E_0(M, N)$ for all other bases.

Then G and H are ordinal with the major argument M. The index of G is r, for if p_1 is a prime different from p,

$$G(p_1^r, p_1) = \delta_r(p_1^r, p_1) = 1 \text{ and } p_1^{r+1} > p_1^r. \quad \text{(See 3.11)}$$

Similarly the index of H is s. But their product is $E_0(M, N)$ which has the index ∞.

8.2. Theorem. *If $f_r(M, N)$ and $\psi_s(M, N)$ are the same as in Theorem 8.1, the index of their composite is r if $s > r$ and $\geqslant r$ if $s = r$.*

Proof. Let $f(M, N) = f_r(M, N) . \psi_s(M, N)$;
$$f_r = O^r(F_1), \psi_s = O^s(F_2).$$

Then
$$\begin{aligned} f(M, N) &= O^r(F_1). O^s(F_2) \\ &= O^r(F_1). O^r[O^{s-r}(F_2)] \\ &= O^r(F_1). O^r(F_3) \\ &= O^r(F_1. F_3) \text{ by (6.21)} \\ &= O^r(F), \text{ where } F = F_1. F_3. \end{aligned}$$

Case 1. If $s > r$, then $t = (s-r) \geqslant 1$ and F_3 has the index t. Hence F being the composite of the non-ordinal function F_1 and the ordinal function F_2, is non-ordinal by (6.33).

Hence $O^r(F)$ has the index r by (3.3).

Case 2. Let $s = r$. Then F is the composite of two non-ordinal functions, and therefore may be *ordinal or non-ordinal*. [E.g. the composite of the two non-ordinal functions $E(M, N) = E(M) E(N)$ and $\phi(M) E(N)$ where ϕ is Euler's function, is $I(M) E(N)$ which is another non-ordinal function; while the composite of $E(M, N)$ with $\phi(M) E^{-1}(N)$, a non-ordinal function, is the ordinal function $I(M) E_0(N)$.] Hence the index of $F \geqslant 0$.

Therefore $f = O^r(F)$ has the index $\geqslant r$.

8.21. As an important corollary of the above theorem we have :—

COR. *The cardinal camponent, $C(M, N)$ of an ordinal unction $f(M, N)$ with the index r and the major argument M, fis ordinal with the precise ordinal index r.*

For by 1.1 $C(M, N) = f(M, N) . f^{-1}(M, 1) E_0(N) . f$ has the index r and $f^{-1}(M, 1)\ E_0(N)$ has the index ∞. Hence by the first part of 8.2 C has the index r.

8.22. From 8.21 we deduce that

The cardinal component $C(M, N)$ of a modular function $F(M, N)$ with the modulus M and the p.m. index r, is identical with that of its primitive and hence is an ordinal function with the index r.

For, if $f(M, N)$ is the primitive of F

$$\begin{aligned} F(M, N) &= f(M, N).\ E_0(M)\, E(N) \\ &= f(M, 1)\, E_0(N).\ C(M, N).\ E_0(M)\, E(N) \\ &= f(M, 1)\ E(N).\ C(M, N) \\ &= F(M, 1)\ E(N).\ C(M, N) \text{ by 7.6.1 } (b). \end{aligned}$$

Hence from 1.1 the result follows.

8.23. *The c-component $C(M, N)$ of $\delta_r(M, N)$ and $d(g_r)$.*

To illustrate 8.21 and 8.22 we will derive the the cardinal component of $\delta_r(M, N)$. By 1.1

$$\begin{aligned} \delta_r(M, N) &= \delta_r(M, 1)\ \delta_r(1, N).\ C(M, N) \\ &= E(M)\, E_0(N).\ C(M, N). \qquad \text{(I)} \end{aligned}$$

$$\begin{aligned} \text{But } \delta^r &= O^r(E) \doteq O^r[E(M)\, E(N)] \\ &= O^r[E(M)\, E_0(N).\ E_0(M)\, E(N)] \\ &= O^r[E(M)\, E_0(N)].\ O^r[E_0(M)\, E(N)] \\ &= E(M)\, E_0(N).\ \text{quasi-princ}_r\, E \qquad \text{(II)} \end{aligned}$$

from 3.3 (ii) and 5.2. Identifying (I) and (II), we see that the cardinal component of δ_r and therefore of $d(g_r)$ also is the quasi-principal function of index r equivalent to $E(N)$.

8.3. THEOREM. *If $f(M, N)$ is an ordinal function with the major argument M and the index r, $f^{-1}(M, N)$ is also ordinal with the index r.*

PROOF. Let $f = O^r(F)$; then $f^{-1} = O^r(F^{-1})$. Because f has the index r, F is non-ordinal. Hence F^{-1} is also non-ordinal, by 6.32. Therefore f^{-1} has the index r.

REMARK. Since $f(M, N).f^{-1}(M, N) = E_0(M, N)$, it follows that the result stated in Th.8.2 for the case $s = r$ is the best possible one. (For, f and f^{-1} have the index r, while E_0 has the index ∞.)

8.4. *The product, the composite and the inverse of quasi-principal functions.*

THEOREM. *If ψ and ψ' are functions of one argument equiavlent to the quasi-principal functions $P_r(M, N)$ $P'_r(M, N)$ with the index r and the major argument M then*

(*a*) (quasi-princ$_r$ ψ)$\times$(quasi-princ$_r$ ψ') = quasi-princ$_r$ $(\psi\times\psi')$

(*b*) quasi-princ$_r$ ψ. quasi-princ$_r\psi'$ = quasi-princ$_r$ $(\psi.\psi')$

(*c*) (quasi-princ$_r$ $\psi)^{-1}$ = quasi-princ$_r$ (ψ^{-1}).

PROOF. (*a*) follows easily from the definition of a quasi-principal function of index r. To prove (*b*), let $P_r = O^{r-1}(P_1)$, $P'_r = O^{r-1}(P_2)$, where P_1 and P_2 are principal functions equiavalent to ψ, ψ' respectively (see. 4).

Then $P_r.\ P'_r = O^{r-1}(P_1).\ O^{r-1}(P_2) = O^{r-1}(P_1.\ P_2) = O^{r-1}(P)$ where*

$$P = P_1.P_2 = \text{princ } \psi.\psi'.$$

Hence quasi-princ$_r$ ψ. quasi-princ$_r$ $\psi' \doteq$ quasi-princ$_r$ $\psi.\psi'$.

To prove (*c*) we have

$$[O^{r-1}(P_1)]^{-1} = O^{r-1}[(P_r)^{-1}] \text{ by } (6.3.1)$$
$$= O^{r-1}(\text{princ } \psi)^{-1}$$

But $(\text{princ } \psi)^{-1} = \text{princ } \psi^{-1}$

Hence (*c*) follows.

* See (1), p. 627.

9. The parallel theorems for modular functions.

From 7.61 (e) and (f) it follows that there can be no theorem parallel to 8.2 and 8.3 in the case of modular functions. Corresponding to 8.1, we have the following

Theorem. *If $F_r(M, N)$ and $\psi_s(M, N)$ are modular functions with the modulus M and the modular indices r, s respectively, their product is a modular function with the modulus M and the precise modular index $\geqslant r$, where r is the smaller of r, s.*

Proof: Let $F(M, N) = F_r(M, N) \times \psi_s(M, N)$; $r \leqslant s$ and M_r, M_s be the effective moduli of F_r, F_s respectively. Then $M_r \geqslant M_s$ and $M_s | M_r$. Hence

$$F(M, N+\lambda M_r) = F_r(M, N+\lambda M_r)\psi_s(M, N+\lambda M_r)$$
$$= F_r(M, N)\psi_s(M, N+\lambda' M_s), \text{ where } \lambda' = \frac{M_r}{M_s}\lambda$$
$$= F_r(M, N)\psi_s(M, N)$$
$$= F(M, N).$$

Thus F has the modular index $\geqslant r$.

Remark. The result stated is the best possible as is shown by the following example:—

Let $F_r(M, N) = E_0(M)E(N)$ for the prime base p
and $= g_r$ for all other bases; $g_r = (M_r, N)$
and $\psi_s(M, N) = g_r$ for the prime base p
and $= E_0(M)E(N)$ for all other bases.

Then F_r and ψ_s are modular functions with the p.m. indices r, s respectively. But their product $[= E_1(M) E(N)]$ by 7.62 has the modular index ∞.

10. Generalization of functions of Ramanujan's class and properties.

Definition. *If the cardinal component $C(M, N)$ of a modular function $F(M, N)$ with the modulus M and the modular index r, is a quasi-principal function of index r with the major argument M, then F is said to be a function of Ramanujan's class with the index r or biefly the class R_r.*

10.1. It follows that, if $C(M, N) = \text{quasi princ}_r\psi$, then, $F(M, N) = F(M, 1)\, E(N).\ \text{quasi-princ}_r\psi$.

We note that Ramanujan's function $C_M(N)$ belongs to the class R_1, while the typical ordinal function $\delta_r(M, N)$ belongs to R_r by 8.23.

10.11. *Generation of the class R_r.* If in the ordinal correspondence we take $F(M, N) = \psi_1(M)\ \psi_2(N)$, then $O^r(F) = f_r(M, N) = \psi_1 \dfrac{M}{N^r} \psi_2(N)$ if $N^r | M$ and 0 otherwise.

The integral $F_r(M, N)$ of f_r w.r.t. the minor argument N is therefore, given by

$$
\begin{aligned}
F_r(M, N) &= \sum_{d^r/M,\ d/N} \psi_1\left(\frac{M}{d^r}\right)\psi_2(d) \\
&= \psi_1(M)\, E(N).\ \text{quasi-princ}_r\psi_2 \\
&= f_r(M, 1) E(N).\ \text{quasi-princ}_r\psi_2 \\
&= F_r(M, 1)\, E(N).\ \text{quasi-princ}_r\psi_2
\end{aligned}
$$

[See 7.61. (*b*)].

Thus we see that if ψ_1, ψ_2 are two variable multiplicative functions of single argument, the integral of $O^r[\psi_1(M)\ \psi_2(N)]$ w.r.t. N generates the class R_r.

10.2. *Properties of the class R_r.* It is clear from 7.61 (*e*) and (*f*), that the composite of two functions of the class R_r and the inverse of any function of R_r do not belong to R_r. Regarding product we have the following

10.3. THEOREM. *The product of a function F of the class R_r and any linear function $\theta(M)$ belongs to R_r.*

PROOF. Let

$F_1(M, N) = F(M, N)\ \theta(M)$; then

$$F_1(M, 1) = F(M, 1)\ \theta(M);\ F_1(1, N) = F(1, N) = E(N)$$

$$
\begin{aligned}
F_1(M, N) &= [F(M, 1)\, E(N).\ \text{quasi-princ}_r\psi]\ \theta(M) \\
&= \theta(M)\ \ F(M, 1)\ \ E\ (N).\ \ \theta(M)\ \text{quasi-princ}_r\psi
\end{aligned}
$$

[See (1), p. 596]

$$= F_1(M, 1)\ E(N).\ \text{quasi-princ}_r\psi(N)\ \theta(N^r).$$

Thus the c-component of $F_1(M, N)$ is a quasi-principal function of index r and hence F_1 belongs to the class R_r.

10.4. THEOREM. *The composite of $F(M, N)$ of class R_r and any quasi-principal function af index r, belongs to R_r..*

PROOF.

$F_1 = F(M, N)$. quasi-princ$_r\psi' = F(M, 1)\ \ E(N)$. quasi-princ$_r\psi$. quas -princ-$_r\psi'$

or $F_1(M, N) = F_1(M, 1)\ E(N)$. quasi-princ$_r\psi.\psi'$ by 8.4 (*b*) [since $F_1(M, 1) = D_M(F_1) = D_N(F)$. D_M(quasi-princ$_r\psi'$) $= F(M, 1)$. $E_0(N) = F(M, 1)$ and similarly, $F_1(1, N) = E(N)$.]

Therefore the theorem follows.

COR. *$\rho_r(M, N)$ belongs to R_r.*

For $\rho(M, N) = E_0(g) = \sum_{d/g} \mu(d) = E(M)\ \ E(N)$. princ E^{-1}

$= E(M, N)$. princ E^{-1}.

Hence $\rho_r = O^r(\rho) = O^r(E). O^r$ (princ E^{-1})

$= \delta_r(M, N)$ quasi-princ$_r\mu(N)$.

Therefore since we know that δ_r belongs to R_r, ρ_r by the above theorem also belongs to it.

10.5. THEOREM. *The composite of a functivn $F(M, N)$ of R_r and any ordinal function with the index infinity belongs to R_r.*

PROOF. We have seen (3.2) that the only ordinal functions with the index ∞ are of the form $\psi(M)\,E_0(N)$. Now let $F_1(M, N) = F(M, N).\ \psi(M)\,E_0(N)$. Then $F_1(M, 1) = F(M, 1).\ \psi(M)$ and $F_1(1, N) = E(N).E_0(N) = E(N)$.

Hence

$$F_1(M, N) = \psi(M)\,E_0(N).\ F(M, 1)\ \ E(N).\ \text{quasi-princ}_r\psi$$
$$= [F(M, 1).\ \psi(M)]\ E(N)\ \text{quasi-princ}_r\psi$$
$$= F_1(M, 1)\,F_1(1, N).\ \text{quasi-princ}_r\psi.$$

Hence F_1 belongs to R.

10.6. THEOREM. *Any multiplicative function $\psi(g_r)$ of $g_r = (M_r, N)$, where M_r = the greatest integer whose rth power divides M, belongs to R_r.*

PROOF. Let $F(M, N) = \psi(g_r)$. Then $F(M, 1) = E(M)$; $F(1, N) = E(N)$ and we know that (see 7) $\psi(g_r)$ has the p.m. index $\leqslant r$. Now the c-component $C(M, N)$ of ψ is from 1.1 given by, $C(M, N) = F(M, N) . \mu(M) \mu(N)$ where $C(M, N) = 0$ if $N^r \times M$ (i.e.) if $N^r > M$. We show that $C(M, N) = 0$ if $N^r < M$. Since C is multiplicative it is sufficient if we show that $C(p^m, p^n) = 0$ if $p^{nr} < p^m$ for any prime p.

Nor

$$C(p^m, p^n) = \sum_{d/p^m, \delta/p^n}\sum F\left(\frac{p^m}{d}, \frac{p^n}{\delta}\right) p(d)\, p(\delta)$$

$$= F(p^m, p^n) - F(p^m, p^{n-1}) - F(p^{m-1}, p^n) + F(p^{m-1}, p^{n-1}).$$

If $p^{nr} < p^m$, $nr < m$. Therefore

$$n < \frac{m}{r} \leqslant \left[\frac{m}{r}\right]; (n-1) < \left[\frac{m}{r}\right].$$

Also $nr \leqslant m-1$ so that

$$n \leqslant \frac{m-1}{r} \leqslant \left[\frac{m-1}{r}\right] \text{ and } n-1 \leqslant \left[\frac{m-1}{r}\right].$$

Hence $C(p^m, p^n) = \psi(p^n) - \psi(p^{n-1}) - \psi(p^n) + \psi(p^{n-1}) = 0$.

Thus $C(M, N) = 0$ if $N^r < M$ as well as when $N^r > M$. Hence it is a quasi-principal function of index r and therefore ψ belongs to R_r.

As an example, we recall here the function $d(g_r)$ whose cardinal component is quasi-princ$_r E$.

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